Towards Systematic Construction of Temporal Logics for Dynamical Systems via Coalgebra

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Agenda

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1. **Introduction**
   - Topic
   - Ingredients
2. **Basic Results**
   - Step Logics: Discrete Time
   - Trajectory Logics: Quantitative Time
   - Orbit Logics: Qualitative Time
3. **Conclusion**

Simple Observations & Naïve Conclusion

1. Dynamical systems exhibit behavior in ***time***
2. Dynamical systems are ***similar*** to automata
3. Automata are coalgebraic
4. Coalgebras induce modal/***temporal*** logics

⇒ Temporal logics for dynamical systems for free?


wikipedia.org

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Binary Relations & Orders

$$R \subseteq X^2$$

Relations Generally

Reflexive	$x \mathrel{R} x$
Transitive	$x \mathrel{R} y \mathrel{R} z \implies x \mathrel{R} z$
Symmetric	$x \mathrel{R} y \implies y \mathrel{R} x$
Antisymmetric	$x \mathrel{R} y \mathrel{R} x \implies x = y$

Orders

Preorder	R+T
Partial Order	R+T+A
Equivalence	R+T+S

Advanced Properties

Non-Branching	$x \mathrel{R} y \wedge x \mathrel{R} z \implies y \mathrel{R} z \vee z \mathrel{R} y$
Linear	$x \mathrel{R} y \vee y \mathrel{R} x$

Monoids

Definition

$$\mathbb{M} = (M, \oplus, e)$$

$$x \oplus e = x = e \oplus x \qquad x \oplus (y \oplus z) = (x \oplus y) \oplus z$$

Induced Order (Left Version)

$$x \leqslant_{\mathbb{M}} z \iff \exists y.\, x \oplus y = z$$

Preorder always

Symmetric iff $\mathbb{M}$ is a group

Linear trivially if $\mathbb{M}$ is a group

Dynamical Systems

Definition

State space S

Time monoid $(T, +, 0)$ of **vectors** (directed differences)
common sense: **durations**

Dynamics $\Phi : S \times T \to S$ right monoid action

$$\Phi(s, 0) = s \qquad \Phi(s, t + u) = \Phi\big(\Phi(s, t), u\big)$$

Derived Concepts

Step $\Phi^t : S \to S$ (transition map)

Trajectory $\Phi_s : T \to S$ (parametric curve)

Orbit $\Phi^\circ : S \to \mathcal{P}S$ (locus map)
$\Phi^\circ(s) = \operatorname{Ran} \Phi_s$

Interpretation

CAVEAT

Dynamical systems look formally like transition systems ***but*** the interpretation is different:

Structure richer

- **Time** density, completeness
- **Space** topology, metric, differential geometry, measure

Behavior weaker

- **deterministic** no arbitrary choice
- **non-pointed** no distinguished initial state
- **total** no spontaneous termination

Examples

Some Notions of Time

T	$\oplus$	e		
$\mathbb{N}$	$+$	0	standard discrete time	irreversible
$\mathbb{Z}$	$+$	0	standard discrete time	reversible
$\mathbb{R}_{\geqslant 0}$	$+$	0	standard continuous time	irreversible
$\mathbb{R}$	$+$	0	standard continuous time	reversible
Σ^*	$\cdot$	ε	automaton input	irreversible

Modal Propositional Logics à la Kripke

States related worlds $(W, \mathrm{R} \subseteq W^2)$

Modalities dual $\Box, \Diamond : \mathcal{L} \to \mathcal{L}$

Satisfaction

$$w \Vdash \Box A \iff v \Vdash A \text{ whenever } w \mathrel{\mathrm{R}} v$$

Expressivity Characterize states up to bisimilarity

- technical stuff (finitary vs. infinitary) aside

Modal Propositional Logics à la Moss

States F-coalgebra $(X, f : X \to FX)$

Modalities universal $\nabla : F\mathcal{L} \to \mathcal{L}$

Satisfaction

$$x \Vdash \nabla\widehat{A} \iff f(x)\ F[\Vdash]\ \widehat{A}$$

Expressivity Characterize states up to bisimilarity

- technical stuff (finitary vs. infinitary) aside

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Step Coalgebras

For any $t \in T$, $(S, \Phi^t : S \to S)$ is an Id-coalgebra.

Modality $\nabla : \mathcal{L} \to \mathcal{L}$

Satisfaction

$$s \Vdash \nabla A \iff \Phi(s, t) \Vdash A$$

Intuition Time-travelling propositions

Example: Next

- Monoid $\mathbb{N}$ generated by element 1.
- Dynamics Φ determined by step Φ^1.
- Modality $\bigcirc$ (*next*) for discrete, irreversible time.

Multistep Coalgebras

For any (finite) subset $U \subseteq T$, $(S, s \mapsto \Phi_s|_U : S \to S^U)$ is a $\mathrm{Hom}(U, -)$-coalgebra.

Modality $\nabla : \mathcal{L}^U \to \mathcal{L}$

Satisfaction

$$s \Vdash \nabla\hat{A} \iff \Phi(s,t) \Vdash \hat{A}(t) \text{ for all } t \in U$$

Projection

$$\bigcirc_t A = \nabla u \mapsto \begin{cases} A & (t = u) \\ \top & (t \neq u) \end{cases}$$

Recombination

$$\nabla\hat{A} = \bigwedge_{t \in U} \bigcirc_t \hat{A}(t)$$

Examples

Discrete Reversible Time

- Monoid $\mathbb{Z}$ generated by elements ± 1.
- Dynamics Φ determined by step Φ^1.
  - monoid act property ensures Φ^{-1} inverse
- Modalities $\bigcirc_{\pm 1}$ (*next*/*prev*) for discrete, reversible time.

Actions

- Monoid Σ^* generated by Σ.
- Dynamics Φ determined by multistep $(\Phi^a)_{a \in \Sigma}$.
- Modalities $[a]/\langle a \rangle$ of *deterministic* dynamic logic & modal μ-calculus.

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Trajectory Coalgebras

$(S, s \mapsto \Phi_s = \mathrm{curry}\,\Phi : S \to S^T)$ is a Hom$(T, -)$-coalgebra.

Modality $\nabla : \mathcal{L}^T \to \mathcal{L}$

Satisfaction

$$s \Vdash \nabla\widehat{A} \iff \Phi(s,t) \Vdash \widehat{A}(t) \text{ for all } t \in T$$

Intuition Proposition-coloured timelines

Notation Intensional special cases of $\mathcal{L}^T$

Examples

Linear Antisymmetric (Totally Ordered) Time

$$\mathit{chg}(t, A, B, C) = \nabla u \mapsto \begin{cases} A & (u < t) \\ B & (u = t) \\ C & (u > t) \end{cases}$$

$$\min t.\, A = \mathit{chg}(t, A, A, \top) \qquad \max t.\, A = \mathit{chg}(t, \top, \top, \neg A)$$

$$\min{}' t.\, A = \mathit{chg}(t, A, \top, \top) \qquad \max{}' t.\, A = \mathit{chg}(t, \top, \neg A, \neg A)$$

$$A \;\mathbf{U}\; B = \bigvee_{t \in T} \mathit{chg}(t, A, B, \top)$$

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Orbit Coalgebras

$(S, \Phi^\circ : S \to \mathcal{P}S)$ is a $\mathcal{P}$-coalgebra

- $\Phi^\circ : S \to \mathcal{P}S$ is equivalent to *reachability* relation $(\rightsquigarrow) \subseteq S^2$
- $\mathcal{P}$-coalgebras induce logics equivalent to Kripke's

Lemma (Properties of Reachability $\rightsquigarrow$)

Preorder	*always*
Non-Branching	*if time is linear*
Symmetric	*if time is symmetric*

Orbit Coalgebras

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Definition

Kripke frame is **orbital** iff it arises from some orbit coalgebra

Theorem (Axiomatization)

Normal modal logics à la Lewis are sound and complete

S4 *for all orbital frames*

S4.3 *for non-branching orbital frames*

S5 *for symmetric orbital frames*

Examples

Stationarity

- Let proposition $@\mathfrak{U}$ characterize a region $\mathfrak{U} \subseteq S$
- Then $\mathfrak{U}$ is a **stationary solution** of Φ iff

$$S \Vdash @\mathfrak{U} \Rightarrow \Box @\mathfrak{U}$$

- Cf. alternative definitions

Informal once entered, never left
Algebraic closed under orbits

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Strategy Flashback

1. Dynamical systems as coalgebras, variously

Step	discrete time
Trajectory	quantitative time
Orbit	qualitative time

2. Modalities by Moss
3. Recover well-known (ad-hoc) temporal operators

Outlook: Open Problems

- Nontrivial applications
- Temporal logics from predicate liftings
- Modal μ-calculus
- Continuous time
- Duration calculus
- Measure-theoretic foundations of DS

2014-03-26: Abel Prize awarded to Y. Sinai