Algebraic–Coalgebraic Recursion Theory of History-Dependent Dynamical System Models

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Agenda

1. **Motivation**
   - Harmonic Oscillation
   - Philosophy of Science
   - Advanced Examples
   - Early Conclusions

2. Theory
   - Course-of-Value Iteration
   - State Systems
   - Applications

3. Conclusion

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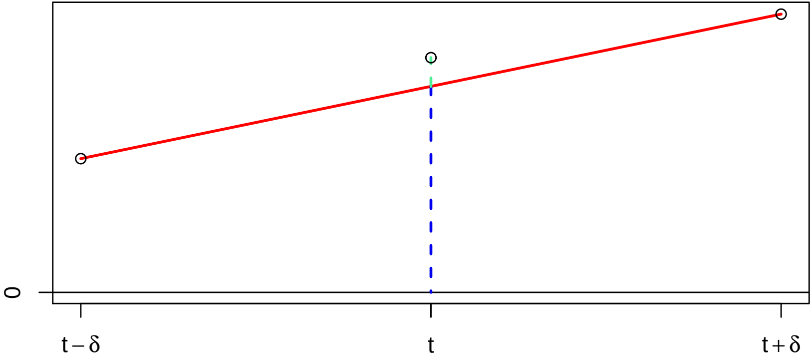
Experimental Setting

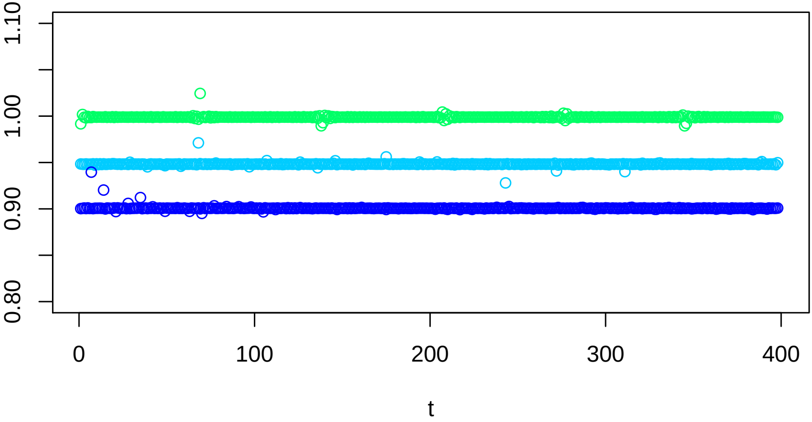
Frictionless point mass	m
acted on by restoring force	
proportional to displacement	x
with coefficient	k
observed regularly with period	δ

Patterns



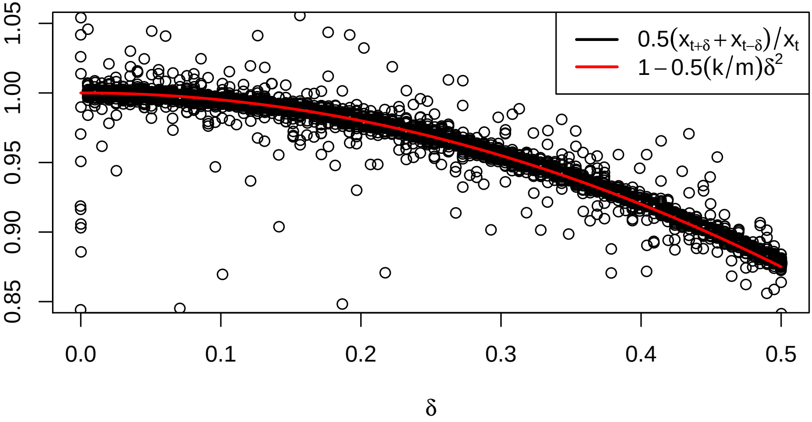
Proportion $\frac{1}{2}(x_{t+\delta} + x_{t-\delta})/x_t = C$

Invariants



Prediction $x_{t+\delta} = 2Cx_t - x_{t-\delta}$ (empirical)

Laws



Prediction $$x_{t+\delta} = \left(2 - \frac{k}{m}\delta^2\right)x_t - x_{t-\delta}$$ (theoretical)

Models

1. $$x_{t+\delta} \approx \left(2 - (\omega\delta)^2\right) x_t - x_{t-\delta}$$
2. $$\begin{pmatrix} x_{t+\delta} \\ v_{t+\delta} \end{pmatrix} \approx \begin{pmatrix} 1 & \delta \\ -\omega^2\delta & 1 \end{pmatrix} \begin{pmatrix} x_t \\ v_t \end{pmatrix}$$
3. $$\begin{pmatrix} x_{t+\delta} \\ v_{t+\delta} \end{pmatrix} = \begin{pmatrix} \cos(\omega\delta) & \omega^{-1}\sin(\omega\delta) \\ -\omega\sin(\omega\delta) & \cos(\omega\delta) \end{pmatrix} \begin{pmatrix} x_t \\ v_t \end{pmatrix}$$
4. $$\ddot{x} = -kx$$

- Reversal of roles: ideal – approximation
- Backward instantiation easy – forward abstraction hard
- Models explain the future in terms of the present **and the past?**

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Questions

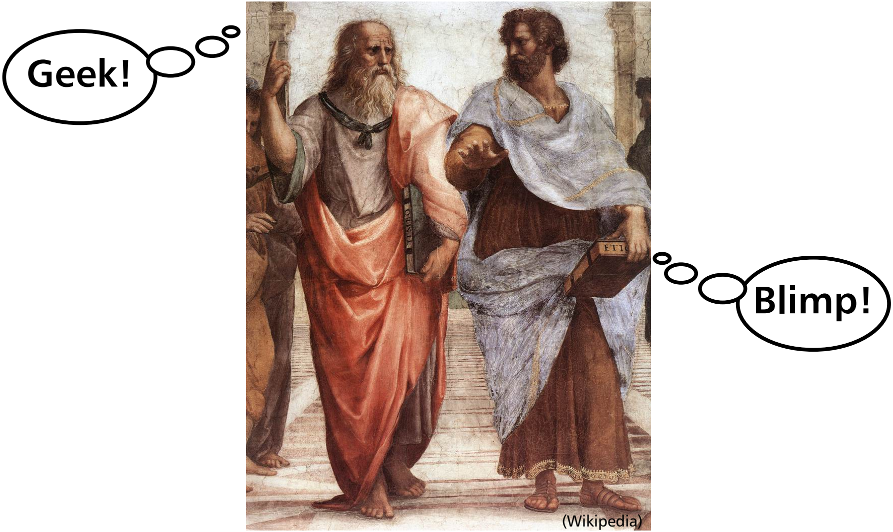
- Is instantaneous state sufficient to “model away” history?
  - If so, what **are** history-dependent models?
  - Success story of classical physics (CP)
- Is the paradigm of CP sufficient for other fields?
  - If so, is it necessary?

Warning

Answers need to be given **simultaneously** in

Mathematics	recursion theory	(coalgebraically!)
Philosophy	epistemology	

Language Issues are Hard to Avoid



(Wikipedia)

Arguments

Contra History (Reductionism)

Laplace's Daemon	History is redundant
Occam's Razor	Redundancy is evil

Pro History (Holism)

Giveaway Prized data sets in "complex systems sciences" are time series not maps

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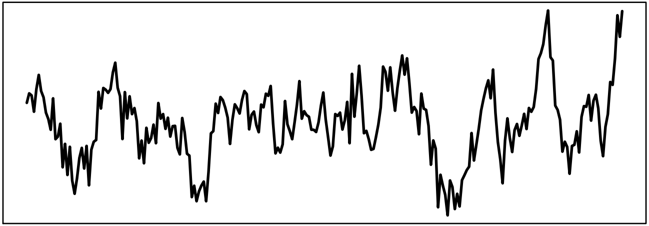
Auto-Regressive Moving-Average (ARMA)
(Box and Jenkins 1970)

- Black-box discrete-time system with I/O
- Output is linear combination of past input **& output**
    - adds controlled autocorrelation to (random) input
- Immensely popular empirical model
    - signal compression
    - statistical forecasting



Trace Function Method (TFM)

(Quinn, Vilkomir, Parnas, and Kostic 2006)

- Black-box event-timed systems with I/O
- Define valid current outputs for trace (history of events)
  - recursive definition of valid traces
- Proposed by Parnas shortly before retirement
  - formal specification & design
  - formal documentation

$$\text{machine}(x_0; (x_1, y_1), (x_2, y_2), \dots) = \begin{cases} \textit{tea!} & y_{1,2,3} = \textit{coffee?} \\ x_0 & \text{otherwise} \end{cases}$$

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Early Conclusions

- Not all modelling is like CP
- More arguments pro history:

ARMA – Empirical Science

Empirical coefficients of history can be inferred
nice analytical theory of inverse modelling

Predictive practically useful forecasting tool

TFM – Software Engineering

Doctrine The "what" not the "how"

Rationale Black-box specification is more durable $\Rightarrow$ valuable than state-based implementation

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Algebraic–Coalgebraic Recursion Theory

Category	some distributive	(**Set**)
Endofunctors	fairly well-behaved	(polynomial)
Initial	algebras $(\mu F, \mathrm{in}_F)$	
Final	coalgebras $(\nu F, \mathrm{out}_F)$	

- Handy abbreviation: C for product functor $C \times (-)$

Ordinary Iteration (Catamorphism)
(Uustalu and Vene 1999)

Rule $\varphi : FC \to C \implies (\!|\varphi|\!)_F : \mu F \to C$

Intuition

- Solution for **elementary** questions
- parameterized with **immediate** subanswers
- gives solution for **well-founded complex** questions.

Universal $h = (\!|\varphi|\!)_F \iff h \circ \mathsf{in}_F = \varphi \circ Fh$

Course-of-Value Iteration (Histomorphism)
(Uustalu and Vene 1999)

Rule $\varphi : F\nu CF \to C \implies \{\varphi\}_F : \mu F \to C$

Intuition
- Solution for **non-well-founded complex** questions
- parameterized with **transitive** subanswers
- gives solution for **well-founded complex** questions.

Universal $h = \{\varphi\}_F \iff h \circ in_F = \varphi \circ F[\langle h, in_F^{-1} \rangle]_{CF}$

Equivalence
$$\{\varphi\}_F = \underbrace{\pi_1 \circ out_{CF}}_{top_{CF}} \circ (\!| \underbrace{out_{CF}^{-1} \circ \langle \varphi, id_{F\nu CF} \rangle}_{\overline{\varphi}} |\!)_F$$
$$(\!| \varphi |\!)_F = \{\varphi \circ F\, top_{CF}\}_F$$

Example Functor

$$N = 1 + (-)$$

$$\mu N = \mathbb{N} \qquad \nu CN = C^{1..\infty} \qquad out_{CN} = cons_C^{-1}$$

$$F\nu CN = C^{0..\infty} \qquad top_{CN} = head_C$$

$$\{\varphi\}_N(0) = \varphi()$$
$$\{\varphi\}_N(1) = \varphi(\{\varphi\}_N(0))$$
$$\{\varphi\}_N(2) = \varphi(\{\varphi\}_N(1), \{\varphi\}_N(0))$$
$$\vdots$$

- Suffices for discrete-time dynamical systems
- but not (easily) for ARMA, TFM, ...

Example Operations

$$\varphi() = 0$$
$$\varphi(a) = a + 1$$
$$\varphi(a, b, \dots) = a + b$$

Fibonacci

Bounded history

$$\varphi(a_1, \dots, a_n) = \sum_{k=0}^{n} \binom{n}{k} a_k$$

Bell

Unbounded history

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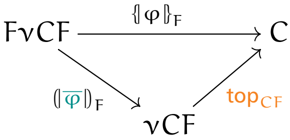
Idea

- Decomposition

  Accumulate $\overline{\varphi}$
  Iterate $(\!|\ |\!)_F$
  Project out_{CF}



- Abstraction
  1. from projection
  2. from accumulation
- Categorification
  - initial objects
  - final objects

Abstract State System

Definition (Abstract (C, F)-Systems)

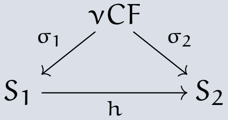
Objects Coslices $(S, \sigma : \nu CF \to S)$ s.t.

$$\exists(\psi : S \to C)\,.\mathrm{top}_{CF} = \psi \circ \sigma$$

State Space S **Abstraction** σ **Valuation** ψ

epic iff σ epic

Morphisms obvious



Lemma

Systems epic $\implies$ ψ *unique;* h *unique & epic*

Initial $(\nu CF, \mathrm{id}) \implies \psi = \mathrm{top}$

Final $(C, \mathrm{top}) \implies \psi = \mathrm{id}$ (epic only!)

Concrete State System

Definition (Concrete φ-Systems)

Objects ... F-algebras $(S, \sigma, \tau : FS \to S)$ s.t. $\tau \circ F\sigma = \sigma \circ \overline{\varphi}$

Transition τ

Morphisms obvious

$$\begin{array}{ccc} FS_1 & \xrightarrow{Fh} & FS_2 \\ \tau_1 \downarrow & & \downarrow \tau_2 \\ S_1 & \xrightarrow[h]{} & S_2 \end{array} \quad (*)$$

Lemma

Systems epic $\Longrightarrow$ *condition [*] redundant*

Initial $(\nu CF, \mathrm{id}, \overline{\varphi})$

Final ??

Simulation

Theorem

If (S, σ, τ) *is a concrete* φ*-system with* ψ *then*

$$\{\!|\varphi|\!\}_F = \psi \circ (\!|\tau|\!)_F$$

Initial state is history at face value

Goal state with instantaneous interpretation
(CP: finite product of measurable variables)

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Example Functor Revisited

Concrete N-systems $(S, \sigma, \tau = [\varepsilon, \triangleright])$

Abstraction $\sigma : C^{1..\infty} \to S$

Initialization $\varepsilon \in S$

Transition $\triangleright : S \to S$

FIFO systems

Definition (Concrete $\mathbb{N}$-State System is ...)

k-bounded $\varphi = \widehat{\varphi} \circ \mathsf{take}(k)$

k-regular $\varphi = \widehat{\varphi} \circ \mathsf{take}(k) \circ \mathsf{append}(\mathfrak{h})$

$$\widehat{\varphi}(a, b) = a + b \qquad \mathfrak{h} = (1, -1)$$

Theorem

FIFO buffer $(C^k, \widehat{\varphi}, \dots)$ *is concrete for* k*-bounded* φ.

Theorem

Epic FIFO systems are final.

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State System Categories (Epic)

Initial	Syntax; history at face value redundant & computationally inefficient
Between	Sweet spot; compressed but regular?
Final	Semantics; fully abstract hard to find; complicated structure Some cases constructible (e.g. FIFO)
Relationship	Morphisms (unique)

- History represents semantic state
- Redundancy harmless
- Quotient structure = equational theory?

Those who cannot remember the past
are condemned to repeat it

J.A.N. Ruiz de Santayana

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