How to Kill Epsilons with a Dagger

A Coalgebraic Take on Systems with Algebraic Label Structure

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Outline

Motivation

Coalgebraic trace semantics for systems with internal (unobservable) behavior is often problematic:

- automata with ε-transitions;
- weak bisimilarity;
- logic programming;
- …

In this work

Abstract framework where:

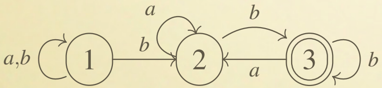
- coalgebras (possibly with internal moves) are modeled as systems of mutually recursive equations;
- trace semantics and a sound ε-elimination procedure are defined as (different) ways of solving systems of equations, using the theory of Elgot Monads.

Trace semantics of NDAs

[*Hasuo, Jacobs & Sokolova, LMCS'07*]

$$\begin{aligned} H\colon\ \mathcal{K}\ell(\mathcal{P}) &\to \mathcal{K}\ell(\mathcal{P}) \\ X &\mapsto (A \times X) + 1 \end{aligned}$$

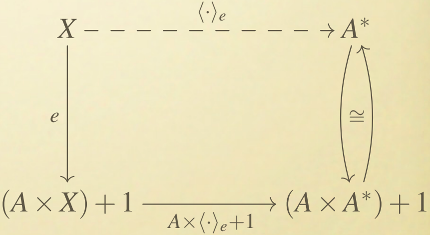
A coalgebra $e\colon X \to H(X)$ is a function $X \to \mathcal{P}((A \times X) + 1)$



$e(1) = \{\langle a,1\rangle, \langle b,1\rangle, \langle b,2\rangle\}$

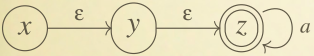
$e(2) = \{\langle a,2\rangle, \langle b,3\rangle\}$

$e(3) = \{\checkmark, \langle a,2\rangle, \langle b,3\rangle\}$



NDAs with ε-transitions

$$
\begin{aligned}
Id + H\colon \quad \mathcal{K}\ell(\mathcal{P}) &\to \mathcal{K}\ell(\mathcal{P}) \\
X &\mapsto X + (A \times X) + 1
\end{aligned}
$$



$$
\begin{array}{ccc}
X & \xrightarrow{\langle\cdot\rangle_e} & (A+1)^* \\
\downarrow e & & \updownarrow \cong \\
X+(A\times X)+1 & \xrightarrow[\langle\cdot\rangle_e+(A\times\langle\cdot\rangle_e)+1]{} & (A+1)^*+(A\times(A+1)^*)+1
\end{array}
$$

$$
\begin{aligned}
e(x) &= \{y\} & \langle x\rangle_e &= \varepsilon\varepsilon a^* \\
e(y) &= \{z\} & \langle y\rangle_e &= \varepsilon a^* \\
e(z) &= \{(a,z),\checkmark\} & \langle z\rangle_e &= a^*
\end{aligned}
$$

internal transitions
are visible!

Algebraic perspective

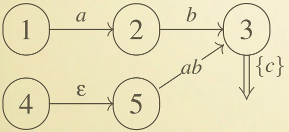
- labels of transitions form a monoid and the ε-transition are those labeled with the unit of the monoid.
- the traditional coalgebraic approach fails because it does not take into account the **algebraic structure** on the labels.

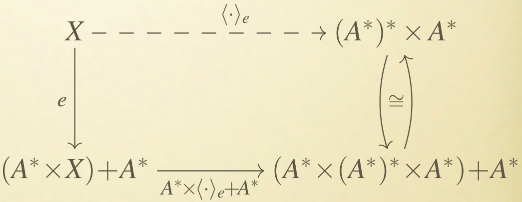
Word automata

$$H\colon \quad \mathcal{K}\ell(\mathcal{P}) \to \mathcal{K}\ell(\mathcal{P}) \qquad X \mapsto (A \times X) + 1$$

$\xrightarrow{\text{Free monad}}$

$$T\colon \quad \mathcal{K}\ell(\mathcal{P}) \to \mathcal{K}\ell(\mathcal{P}) \qquad X \mapsto (A^* \times X) + A^*$$





$$\langle 1\rangle_e = \{[a,b,c]\} \quad \neq \quad \langle 4\rangle_e = \{[\varepsilon, ab, c]\}$$

NDA (with or without ε-transitions) can be interpreted as word automata.

Elgot monads

$$(T, \eta, \mu, (\cdot)^\dagger) \qquad\qquad \frac{e\colon X \to T(X+Y)}{e^\dagger\colon X \to T(Y)}$$

Intuitively:

- ◦ X is a set of variables;
- ◦ Y is a set of parameters;
- ◦ $e\colon X \to T(X+Y)$ is a system of (mutually recursive equations);
- ◦ $e^\dagger\colon X \to T(Y)$ is a substitution solving the system e.

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & TY \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu^T_Y} \\ T(X+Y) & \xrightarrow[T[e^\dagger, \eta^T_Y]]{} & TTY \end{array}$$

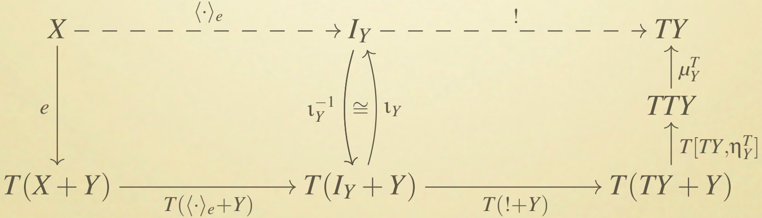
$(\cdot)^\dagger$ satisfies certain axioms...

$e^\dagger$ as "canonical fixpoint" solution

$(T, \eta, \mu, (\cdot)^\dagger)$

$$\frac{e\colon X \to T(X+Y)}{e^\dagger\colon X \to T(Y)}$$

C a **Cppo**-enriched category.
$T\colon C \to C$ locally continuous.



$e^\dagger$ as “canonical fixpoint” solution

$(T, \eta, \mu, (\cdot)^\dagger)$

$$\frac{e : X \to T(X+Y)}{e^\dagger := (! \circ \langle\cdot\rangle_e) : X \to T(Y)}$$

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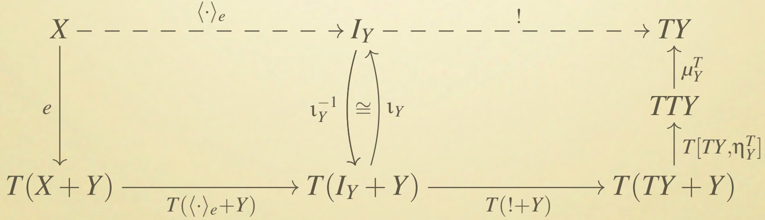

$e^\dagger$ as "canonical fixpoint" solution

$$\begin{array}{rll} T\colon & \mathcal{K}\ell(\mathcal{P}) & \to \mathcal{K}\ell(\mathcal{P}) \\ & X & \mapsto (A^* \times X) + A^* \end{array}$$

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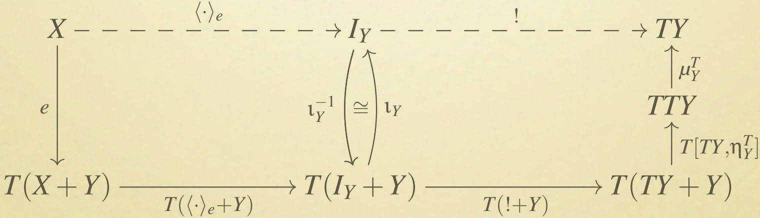
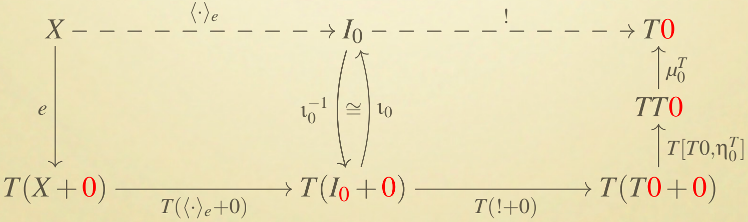


$e^\dagger$ as "canonical fixpoint" solution

$$
\begin{aligned}
T:\quad \mathcal{K}\ell(\mathcal{P}) &\to \mathcal{K}\ell(\mathcal{P})\\
X &\mapsto (A^* \times X) + A^*
\end{aligned}
\qquad\qquad
\frac{e: X \to T(X+0)}{e^\dagger := (!\circ\langle\cdot\rangle_e): X \to T(0)}
$$

C a **Cppo**-enriched category.
$T: C \to C$ locally continuous.

Set parameter $Y = 0$.



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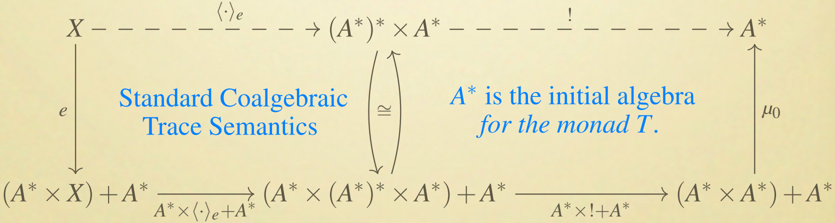

Trace semantics of word automata

$$T: \quad \mathcal{K}\ell(\mathcal{P}) \to \mathcal{K}\ell(\mathcal{P})$$
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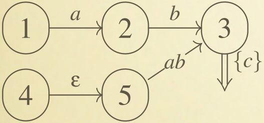
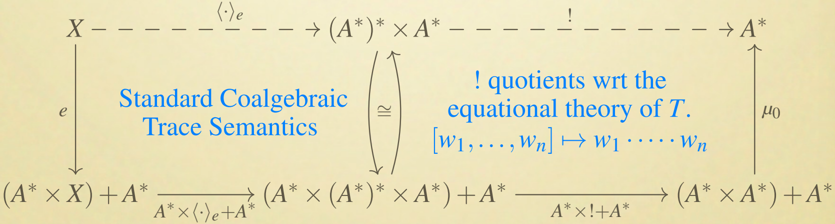


$$\langle 1 \rangle_e = \{[a,b,c]\} \qquad e^\dagger(1) = !(\langle 1 \rangle_e) = \{abc\}$$
$$\langle 4 \rangle_e = \{[\varepsilon,ab,c]\} \qquad e^\dagger(4) = !(\langle 4 \rangle_e) = \{abc\}$$



The framework

Uniform trace semantics (using † of the monad T)

- For $e\colon X \to TX$ a word automaton:

$$[\![\cdot]\!]_e \coloneqq e^{\dagger}\colon X \to A^*.$$

- For $e\colon X \to HX$ an NDA:

$$[\![\cdot]\!]_e \coloneqq (\kappa_X \circ e)^{\dagger}\colon X \to A^*.$$

  where $\kappa\colon H \Rightarrow T$ is the universal map of the free monad T on H.

- For $e\colon X \to X + HX$ an NDA with ε-transitions:

$$[\![\cdot]\!]_e \coloneqq ([\eta_X, \kappa_X] \circ e)^{\dagger}\colon X \to A^*$$

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ε-elimination (using † of the exception monad $Id + HX$)

$$\big(e\colon X \to X + HX\big) \;\mapsto\; \big(e\backslash\varepsilon\colon X \to HX\big)$$

ε-elimination

$Id + HX\colon \quad \mathcal{K}\ell(\mathcal{P}) \;\to \mathcal{K}\ell(\mathcal{P})$

$$\frac{e\colon X \to X{+}Y{+}HX}{e^{\dagger} \coloneqq (!\circ\langle\cdot\rangle_e)\colon X \to Y{+}HX}$$

ε-elimination

$$Id + HX\colon\quad \mathcal{K}\ell(\mathcal{P}) \to \mathcal{K}\ell(\mathcal{P}) \qquad\qquad \frac{e\colon X \to X + HX}{e^\dagger \coloneqq (! \circ \langle\cdot\rangle_e)\colon X \to HX}$$

Set parameter $Y = 0$:

$$\begin{array}{ccccc}
X & \xrightarrow{\langle\cdot\rangle_e} & \mathbb{N} \times HX & \xrightarrow{!} & HX \\
\Big\downarrow e & & \cong & & \Big\uparrow \mu_0 = \nabla \\
X + HX & \xrightarrow[\langle\cdot\rangle_e + HX]{} & (\mathbb{N} \times HX) + HX & \xrightarrow[! + HX]{} & HX + HX
\end{array}$$

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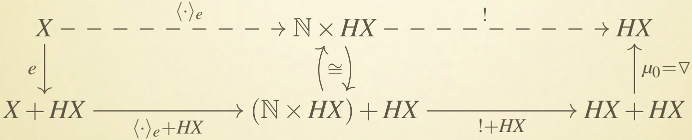

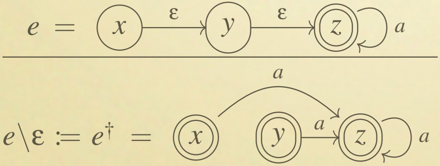

$$\begin{aligned}
\langle x\rangle_e &= \{(2,a,z),(2,\checkmark)\}\\
\langle y\rangle_e &= \{(1,a,z),(1,\checkmark)\}\\
\langle z\rangle_e &= \{(0,a,z),(0,\checkmark)\}\\
e\backslash\varepsilon(x) &= \{(a,z),\checkmark\}\\
e\backslash\varepsilon(y) &= \{(a,z),\checkmark\}\\
e\backslash\varepsilon(x) &= \{(a,z),\checkmark\}
\end{aligned}$$

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Soundness of ε-elimination

$$[\![\cdot]\!]_e \;=\; [\![\cdot]\!]_{e\backslash\varepsilon}$$

Discussion

- In a sense, our framework encompasses [Hasuo, Jacobs & Sokolova, LMCS'07]. We have slightly more restrictive assumptions: local continuity in place of local monotonicity.
- ε-elimination is like in [Silva & Westerbaan, CALCO'13], but in a more abstract setting with more instances.
- Other features of our framework: the algebra of labels does not need to be free - e.g. Mazurkiewicz traces (in the paper).
- Open question:
  - assumptions of the framework: initial algebra-final coalgebra coincidence and an equational property (the double dagger law) of †. How to formulate it without any **Cppo**-enrichment?