Fixed Points of Functors

CMCS 2016

Dedicated to the memory of Václav Koubek

Initial Algebras and Terminal Coalgebras

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2017 (?)

FIXED POINTS

$FX = X$
(too much)

$FX \cong X$
(too little)

$FX \xrightarrow{\cong} X$ (an algebra)

$X \xrightarrow{\cong} FX$ (a coalgebra)

- μF initial algebra
- νF terminal coalgebra
- ρF rational fixed point

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Examples

- Automata: $FX = X^{\Sigma} \times \{0,1\}$
  - $\mu F = \emptyset$
  - $\nu F = \mathcal{P}\Sigma^{\star}$ (Manes and Arbib 1986)
  - $\rho F =$rational languages over Σ

- Streams: $FX = X \times \Sigma + 1$
  - $\mu F = \Sigma^{\star}$
  - $\nu F = \Sigma^{\infty}$, finite and infinite streams
  - $\rho F =$finite or eventually periodic streams

Part 1: Initial Algebras

Lawvere 1964: NNO as the initial algebra of the functor $FX = X + 1$

Lambek 1969: initial algebra as a fixed point

Barr 1970: the first use of "algebra for a functor"

polynomial functors

Polynomial functor H_Σ for a signarture Σ

$$H_\Sigma X = \coprod_{\sigma \in \Sigma} X^{ar(\sigma)}$$

initial algebra: all finite Σ-labelled trees

label $\sigma \Rightarrow$ has $ar(\sigma)$ children

ONLY FOR FINITARY SIGNATURES

initial algebra: all well-founded Σ-labelled trees

well-foundedness in general?

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cartesian subcolagebras

Cartesian subcoalgebras of a coalgebra (A, α): the square

$$\begin{array}{ccc} B & \xrightarrow{\beta} & FB \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle Fm} \\ A & \xrightarrow{\alpha} & FA \end{array}$$

is a pullback

Example: $F = \mathcal{P}$. A subcoalgebra B of a graph A:

$$x \text{ lies in } B \Rightarrow \text{all neighbours lie in } B,$$

a cartesian subcoalgebra:

$$x \text{ lies in } B \Leftrightarrow \text{all neighbours lie in } B$$

well-founded coalgebras

A coalgebra is well-founded if it has no proper cartesian subcoalgebra.

Taylor 1995 for functors preserving inverse images. Inspired by:

Example (Osious 1974): For $F = \mathcal{P}$ a graph is well-founded iff it has no infinite paths.

Theorem (Taylor 1995): Let a set functor preserve intersections. A coalgebra (A, α) is well-founded iff its canonical graph $(A, \rightsquigarrow)$ is well-founded:

$$x \rightsquigarrow y \Leftrightarrow y \text{ lies in every subcoalgebra containing } x$$

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well-founded part

Theorem (Adámek, Milius, Moss 2017?): Every coalgebra of a set functor has the largest well-founded subcoalgebra called

the well-founded part.

Graphs: all vertices from which no infinite path starts

Theorem (Adámek, Milius, Moss 2017?): A set functor has an initial algebra iff it has a fixed point. The well-founded part of any fixed point of F is μF.

Theorem (Adámek, Milius, Moss & Sousa 2012): For all set functors

initial algebra=terminal well-founded coalgebra

(For set functors preserving inverse images: Taylor 1995.)

Does not generalize to many-sorted sets:

$$F(X, Y) = (\mathcal{P}X, \emptyset) \text{ if } Y = \emptyset, \text{ else } (1, 1)$$

does not have an initial algebra.
But $(1, 1)$ is a fixed point, indeed, the terminal well-founded coalgebra.

Fixed point does not imply initial algebra!

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recursive coalgebras

Recursive coalgebras (A, α) are those having for every algebra a unique coalgebra-to-algebra homomorphism:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & FA \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle Fh} \\ B & \xleftarrow[\beta]{} & FB \end{array}$$

(Taylor 1999, Capretta, Uustalu and Vene 2006)

Theorem (Taylor 1999): If F preserves monomorphisms

recursive $\Rightarrow$ well-founded.

If F preserves inverse images

recursive $\Leftrightarrow$ well-founded.

initial chain

Definition (Adámek 1974): The initial chain of a functor F is the transfinite chain

$$\emptyset \xrightarrow{\ !\ } F\emptyset \xrightarrow{F!} F^2\emptyset \xrightarrow{F^2!} \cdots$$

Limit steps are defined by chain-colimits, isolated steps by applying F:

$$F^{i+1}\emptyset = F(F^i\emptyset)$$

Fact: each of the coalgebras $F(F^i\emptyset) \to F^i\emptyset$ is well-founded. If this is a fixed point, then it is the initial algebra.

The least such i is called the convergence of the initial chain.

convergence of the initial chain

Polynomial functors: $H_\Sigma X = \coprod_{\sigma \in \Sigma} X^{ar(\sigma)}$

The initial chain converges in λ steps if the signature is λ-ary (λ an infinite regular cardinal).

Theorem (Adámek & Trnková 2011). For every set functor F with μF the initial chain converges in λ steps, where λ is an infinite regular cardinal or $\lambda \leq 3$.

MANY-SORTED SETS

μF exists $\Rightarrow$ initial chain converges.

But the convergence can be any ordinal

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Section 2: Terminal Coalgebras

Scott (1972): Model of untyped λ-calculus

Categorical analysis: Wand 1977, Smyth and Plotkin 1978

Manes and Arbib (1986):

- Streams as a terminal coalgebra of $FX = X^\Sigma + 1$
- Languages as a terminal coalgebra of $FX = X^\Sigma \times \{0, 1\}$

Barr (1993): Terminal cochain, a dualization of the initial chain

$$1 \xleftarrow{\;!\;} F1 \xleftarrow{\;F!\;} F^2 1 \xleftarrow{\;F^2!\;} \cdots$$

Rutten (2000): Universal coalgebra. Terminal coalgebras express the abstract behavior of states of systems (=coalgebras).

fixed points and νF

Theorem (Adámek and Koubek 1990) Assume GCH:
F has a fixed point $\Leftarrow \nu F$ exists $\Leftarrow F$ has a fixed-point pair:

$$F(\lambda) \cong \lambda \text{ and } (F\lambda^+) \cong \lambda^+$$

CHARACTERIZATION OF TERMINAL COALGEBRAS?

There are set functors F and G which coincide on objetcs and

$$\nu F \text{ exists and } \nu G \text{ does not}$$

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accessible functors

F is λ-accessible if it preserves λ-fileterd colimits

For set functors equivalently: every element of FX lies in the image of Fm for some subset $m : M \hookrightarrow X$ with $card M < \lambda$

H_Σ is λ-accessible iff Σ is λ-ary

Theorem (Worrell 2005): For a λ-accessible set functor

- The terminal chain converges in $\lambda + \lambda$ steps.
- The connecting morphisms from λ onwards are monomorphisms.

EASY PROOF

Trnková's cover of F is a functor F^T preserving finite intersections and agreeing with F on non-empty sets and functions

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proof of Worrell's theorem

1. If $F1 = \emptyset$, then $F \equiv \emptyset$ - trivial case.

If $F1 \neq \emptyset$ we conclude that nonempty coalgebras exist, thus the terminal coalgebra cannot be empty. This implies

$$F^i 1 \neq \emptyset$$

for all ordinals i.

2. Thus F and F^T have the same terminal cochain.
From now on we can assume that F preserves finite intersections.

3. The connecting map $F^{\lambda+1}1 \to F^{\lambda}1$ is monic.

4. All connecting maps from λ onwards are monic:
F preserves monomorphisms, and limits of chains of monomorphisms are also monic.

5. We are ready to prove that F preserves the limit

$$F^{\lambda+\lambda}1 = \lim F^{\lambda+i} \text{ for } i < \lambda$$

This is an intersection of a λ-cochain of subobjects

$$v_i : F^{\lambda+i}1 \hookrightarrow F^{\lambda}1$$

(the connecting morphisms of the cochain) for $i < \lambda$.

Given x in $F(F^{\lambda}1)$ lying in the image of each Fv_i we prove that it also lies in the image of Fv_{λ}. Since F is λ-accessible, there exists $m : M \hookrightarrow F^{\lambda}1$ with $cardM < \lambda$ and $x \in im(Fm)$. And F preserves the intersection w_i of m and v_i, thus $x \in im(Fw_i)$. From $cardM < \lambda$ we conclude that there exists i_0 such that w_i is independent of i for all $i > i_0$. This proves that x lies in the image of Fv_{λ}.

examples

- H_Σ: convergence in ω steps, even if Σ is infinitary
- $\mathcal{P}_\lambda$: convergence in $\lambda + \omega$ steps (Worrell 2005)

OTHER LENGTHS OF CONVERGENCE?

- $\mathcal{F}_\lambda$, the filter-functor restricted to λ-small filters, i.e. those containing a member smaller than *lambda*: convergence in $\lambda + \lambda$ steps
- $\mathcal{P}'_\lambda$, the modification of $\mathcal{P}_\lambda$ where $\mathcal{P}'_\lambda f$ sends a set A to $f[A]$ if f restricted to A is monic, else to $\emptyset$: convergence in λ steps

(Adámek, Koubek and Palm, submitted)

well-pointed coalgebras

pointed coalgebra (A, α, a): an element $a \in A$ is chosen

well-pointed means that no proper subcoalgebra and no proper qutient-coalgebra exist

Automata: a pointed coalgebra is a classical deterministic automaton. Well-pointed coalgebra= minimal automaton.

T is the collection of all well-pointed coalgebras
up-to isomorphism

Theorem (Adámek, Milius, Moss & Sousa 2013): A set functor preserving intersections has a terminal coalgebra iff T is a set.

well-pointed coalgebras form νF

Let F preserve intersections.

Every coalgebra (A, α) defines a map $\alpha^+ : A \to T$
sending an element $a \in A$ to the well-pointed coalgebra a defines.

T as a coalgebra: the isomorphism $T \to FT$ assigns to every element (A, α, a) of T the following element of FT:

$$1 \xrightarrow{a} A \xrightarrow{\alpha} FA \xrightarrow{F\alpha^+} FT$$

examples of $T = \nu F$

- Automata, $FX = X^{\Sigma} \times \{0, 1\}$. Here T are all minimal automata, isomorphic to $\mathcal{P}\Sigma^*$.
- Graphs, $F = \mathcal{P}$: Well-pointed graphs yield by tree-expansion strongly extensional trees, i.e. tress with no nontrivial tree bisimilarity (Worrell 2005). Conversely, every strongly extensional tree is the expansion of a well-pointed graph, unique up-to isomorphism:

$$T = \text{all strongly extensional graphs (up to isomorphisms)}$$

- Polynomial functors H_{Σ}. T is the set of all Σ-trees: to every well-pointed coalgebra assign the tree-expansion of the chosen point.

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initial algebras revisited

Theorem (Adámek, Milius, Moss & Sousa 2013): Let a set functor F preserve intersections. Then F has an initial algebra iff the collection I of all well-founded, well-pointed coalgebras is a set (up to isomorphism):

$$\mu F = I \text{ as a subcoalgebra of } T$$

- H_Σ: all well-founded Σ-trees
- $\mathcal{P}_f$: all hereditarily finite sets. For every set the canonical picture is a well-founded, well-pointed graph. Which is finitely branching iff the set is hereditarily finite. Conversely, every finitely branching, well-pointed, well-founded graph is a canonical picture of a unique hereditarily finite set.
- Automata: $I = \emptyset$

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terminal coalgebras as algebras

completely iterative algebra $\alpha : FA \to A$: for every object X (of variables) and every (recursive equation) morphism $e : X \to FX + A$ there exists a unique solution $e^\dagger$:

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [a,A]} \\ FX + A & \xrightarrow[Fe^\dagger + A]{} & FA + A \end{array}$$

Iterative algebra: as above, but only for finitly presentable X of variables.

Theorem (Milius 2005): The terminal coalgebra is precisely the initial completely iterative algebra.

trees and languages

- H_Σ: all Σ-trees form a completely iterative algebra: solutions of recursive equations by tree-expansion

  All rational Σ-trees, i.e. having only finitely many subtrees up-to isomorphism, form an iterative subalgebra.

- Example $FX = X^\Sigma \times \{0,1\}$: all languages over Σ form a completely iterative algebra.

  All rational (=regular) languages form an iterative subalgebra.

Part 3: rational fixed point

Let F be a finitary set functor, i.e. preserving filtered colimits. (Equivalently: every element of FX lies, for some finite subset $m : M \hookrightarrow X$, in the image of Fm.)

The rational fixed point ρF of F is the initial iterative algebra.

Theorem (Adamek, Milius, Velebil 2006): ρF is the colimit of all finite coalgebras for F.

examples of rational fixed points

- Automata: all rational languages
- Polynomial functors: all rational trees
- $\mathcal{P}_f$: all non-well-founded sets whose canonical picture is finite. Comparison:
  - $\nu\mathcal{P}_f =$ all non-well-founded sets whose canonical picture is finitely branching.
  - $\mu\mathcal{P}_f =$ all well-founded sets with a finite canonical picture.
- $FX = \Sigma \times X$, streams: all eventually periodic streams

applications of rational fixed points

Algebraic trees, i.e. solutions of higher-order recursive program schemes (Courcelle 1983)

- rational fixed points in presheaf categories, see Adámek, Milius and Velebil (2010)
- rational fixed points in nominal sets, see Milius, Pattinson and Wissmann (2016).

Another application is a generalization of Eilenberg's result that varieties of rational languages correspond bijectively to pseudovarieties of monoids. In our paper Adámek, Milius, Myers and Urbat 2015 we applied the rational fixed point in categories of

- boolean algebras
- vector spaces
- semilattices

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