Proving Equations on Stream GSOS via Bisimulation on Open Terms

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Stream systems over an alphabet A can be represented as coalgebra for the functor $F = A \times -$ on **Set**. These are couples $(X, \langle o, d\rangle)$ where X is a set of states and $\langle o, d\rangle : X \to A \times X$ is the structure assigning to each state $t \in X$, its output value $o(t)$ and its next state $d(t)$. Operations over streams can be defined by means of stream GSOS-rules [2]. Fixed a signature Σ, a specification induces a system $(T_\Sigma\emptyset, \langle o, d\rangle)$ over the set of closed terms $T_\Sigma\emptyset$. For an example, take Σ to be of a set of constants $\{\mathsf{a} \mid a \in A\}$ together with a binary operation alt: the set of GSOS-rules in Figure 1 defines the semantics of these operators and Figure 2 shows the stream system rechable from the term $\mathsf{alt}(\mathsf{a}, \mathsf{alt}(\mathsf{b}, \mathsf{c})) \in T_\Sigma\emptyset$.

$$\frac{}{\mathsf{a} \xrightarrow{a} \mathsf{a}}\ \forall a \in A \qquad \frac{x \xrightarrow{a} x' \quad y \xrightarrow{b} y'}{\mathsf{alt}(x, y) \xrightarrow{a} \mathsf{alt}(y', x')}\ \forall a, b \in A$$

Fig. 1. The GSOS-rules of our running examples

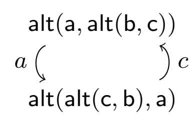

Fig. 2. Stream system

In order to check that the two terms denote the same streams, one can use *bisimulation*: for an arbitrary stream system $\langle o, d\rangle : X \to A \times X$, a relation $R \subseteq X \times X$ is a bisimulation if $(t_1, t_2) \in R$ implies $o(t_1) = o(t_2)$ and $d(t_1)\ R\ d(t_2)$. If t_1 and t_2 are related by a bisimulation, then they are said bisimilar (written $t_1 \sim t_2$) and they denote the same stream. For instance $\mathsf{alt}(\mathsf{a}, \mathsf{alt}(\mathsf{b}, \mathsf{c}))$ and $\mathsf{alt}(\mathsf{a}, \mathsf{alt}(\mathsf{d}, \mathsf{c}))$ are bisimilar: see Figure 2, b does not play any role in $\mathsf{alt}(\mathsf{a}, \mathsf{alt}(\mathsf{b}, \mathsf{c}))$ and, similarly, d does not play any role in $\mathsf{alt}(\mathsf{a}, \mathsf{alt}(\mathsf{d}, \mathsf{c}))$. Indeed this is a particular case of a more general fact: $\mathsf{alt}(t_1, \mathsf{alt}(t_2, t_3)) \sim \mathsf{alt}(t_1, \mathsf{alt}(t_4, t_3))$ for all $t_1, t_2, t_3, t_4 \in T_\Sigma\emptyset$ $(\star)$. To prove this, it is needed to consider an infinite number of bisimulations, one for each instantiation of the terms $t_i's$, or just a bisimulation that includes all possible instantiations of the terms. In both cases, we have to check that the bisimulation conditions are satisfied by infinite number of pairs.

Let $T_\Sigma\mathcal{V}$ be the set of (open) terms of Σ over a set of variables $\mathcal{V}$. We say that two terms $t_1, t_2 \in T_\Sigma\mathcal{V}$ are equivalent, denoted by $t_1 \equiv t_2$, if for every closed substitution $\sigma \in (T_\Sigma\emptyset)^{\mathcal{V}}$, $\sigma(t_1) \sim \sigma(t_2)$. Then, the previous property can be expressed in a compact way by:

$$\mathsf{alt}(X, \mathsf{alt}(Y, Z)) \equiv \mathsf{alt}(X, \mathsf{alt}(W, Z)) \text{ where } X, Y, Z, W \in \mathcal{V}\,.$$

The goal of our research is to define a bisimulation proof technique for equivalence of open terms. A good motivation to study this problem is to avoid dealing with infinite relations when proving properties like $(\star)$.

$$\frac{x \xrightarrow{\sigma|a} x' \quad y \xrightarrow{\sigma|b} y'}{\overline{\mathsf{alt}}(x,y) \xrightarrow{\sigma|a} \overline{\mathsf{alt}}(y',x')} \qquad \frac{}{X \xrightarrow{\sigma|\sigma(X)} X}$$

Fig. 3. $X \in \mathcal{V}, \sigma \in A^{\mathcal{V}}$

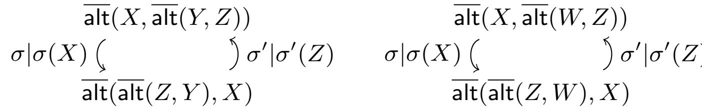

Fig. 4. $\sigma, \sigma' \in A^{\mathcal{V}}$

Our approach is based on pointwise extensions of operations, as studied by Hansen and Klin. In [1], they show that given an operation f on a final F-coalgebra defined by a GSOS specification, it is possible to construct, systematically, another GSOS specification that defines the pointwise extension $\overline{f}$ of f in the final F^B-coalgebra for arbitrary B. The syntax $\overline{\Sigma}$ of the new specification extends the original syntax Σ with auxiliary operators.

In our case, $F = A \times -$ and we set $B = A^{\mathcal{V}}$. Then any stream GSOS specification lifts to a specification on $(A\times -)^{A^{\mathcal{V}}}$-coalgebras, i.e., Mealy machines with inputs $A^{\mathcal{V}}$ and outputs A. For instance, the lifting of the operator alt is $\overline{\mathsf{alt}}$ and it is defined by the first rule in Figure 3 (we omit the lifting of the constants).

We explain the intuition behind the lifting and we introduce the last element needed by our framework (the second rule in Figure 3). If an operation over streams is defined by GSOS-rules then its semantics only depends on the outputs of its parameters. For example, consider $\mathsf{alt}(X, t)$ with $X \in \mathcal{V}$ and $t \in T_\Sigma\mathcal{V}$. If the outputs of (the streams represented by) X and t are fixed, the output of $\mathsf{alt}(X, t)$ is defined. The role of the input substitution $\sigma \in A^{\mathcal{V}}$ is to “fix” the output of the variables and, by induction, the output of an arbitrary open term. The inductive cases are taken into account by the lifting of the rules in the stream specification. The base cases, for $X \in \mathcal{V}$, are defined by the second rule in Figure 3.

The lifting of the rules and the axioms associated to each variable $X \in \mathcal{V}$ define a Mealy machines on open terms $T_{\overline{\Sigma}}\mathcal{V}$. In Figure 4, we depict the Mealy machines associated to $\overline{\mathsf{alt}}(X, \overline{\mathsf{alt}}(Y, Z))$ and $\overline{\mathsf{alt}}(X, \overline{\mathsf{alt}}(W, Z))$. It should be clear that both systems are bisimilar. In addition, notice that it is straightforward to define a finite bisimulation that relates them.

Let $\sim_{\mathcal{M}}$ be the bisimilarity relation on the induced Mealy machine on open terms. We can state, informally, the following result.

Theorem 1. *For all $t_1, t_2 \in T_\Sigma\mathcal{V}$: $t_1 \sim_{\mathcal{M}} t_2$ iff $t_1 \equiv t_2$.*

Our proof uses some peculiarity of stream systems and, indeed, the results seems to not extend to arbitrary coalgebras. However for the case of streams, being able to reduce infinite proofs into finite ones, give us the hope to automate some proofs of equivalence for open terms.

References

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2. Bartek Klin. Bialgebras for structural operational semantics: An introduction. *Theor. Comput. Sci.*, 412(38):5043–5069, 2011.