Behavioural equivalences for coalgebras with unobservable moves

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Motivating example: LTSs with silent moves

Put $\Sigma_\tau = \Sigma + \{\tau\}$.

Definition

A labelled transition system with silent moves is a map:

$$\alpha : X \to \mathcal{P}(\Sigma_\tau \times X) \cong \mathcal{P}(\Sigma \times X + X)$$

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Definition (one of the variants)

Weak bisimulation on $\alpha : X \to \mathcal{P}(\Sigma_\tau \times X) \stackrel{df}{=}$ strong bisimulation on its saturation α^*.

Here, the LTS saturation $\alpha^* : X \to \mathcal{P}(\Sigma_\tau \times X)$ of α is the smallest LTS containing all transitions of α and satisfying:

$$\frac{}{x \xrightarrow{\tau} x} \qquad \frac{x \xrightarrow{a} x' \quad x' \xrightarrow{\tau} x''}{x \xrightarrow{a} x''} \qquad \frac{x \xrightarrow{\tau} x' \quad x' \xrightarrow{a} x''}{x \xrightarrow{a} x''}$$

More systems with silent moves. . .

- Segala systems,
- fully probabilistic systems,
- ...

Coalgebras with internal moves

More systems with silent moves...

- Segala systems,
- fully probabilistic systems,
- ...

$$\frac{\text{Coalgebras with internal moves}}{X \to T(FX + X)}$$

For LTS $T = \mathcal{P}$, $FX = \Sigma \times X$, so

$$T(FX + X) = \mathcal{P}(\Sigma \times X + X) \cong \mathcal{P}(\Sigma_\tau \times X).$$

Hiding internal moves inside a monadic structure

Put $F_\tau = F + \mathcal{I}d$ and consider the functor

$$T(F + \mathcal{I}d) = TF_\tau$$

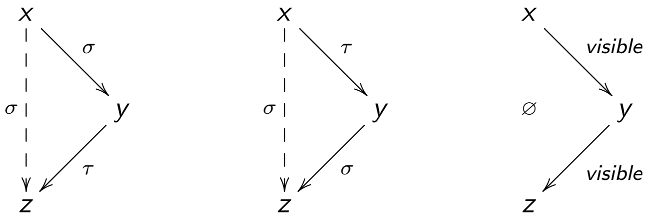
Theorem

If $\mathcal{K}l(T)$ admits zero morphisms (i.e. $0 \cdot f = g \cdot 0 = 0$) and $F : \mathsf{C} \to \mathsf{C}$ lifts to $\mathcal{K}l(T)$ then we may impose a monadic structure on TF_τ.

LTS monad

Put $T = \mathcal{P}$ and $F_\tau = \Sigma_\tau \times \mathcal{I}d$. The LTS functor $\mathcal{P}(\Sigma_\tau \times \mathcal{I}d)$ carries a monadic structure whose composition in the Kleisli category is given as follows. For $f : X \to \mathcal{P}(\Sigma_\tau \times Y)$ and $g : Y \to \mathcal{P}(\Sigma_\tau \times Z)$ we have $g \cdot f : X \to \mathcal{P}(\Sigma_\tau \times Z)$:

$$g{\cdot}f(x) = \{(\sigma, z)) \mid x \xrightarrow{\sigma}_f y \xrightarrow{\tau}_g z \text{ or } x \xrightarrow{\tau}_f y \xrightarrow{\sigma}_g z \text{ for some } y \in Y\}.$$



Section summary

Systems with internal moves

Section summary

$$\frac{\text{Systems with internal moves}}{\text{Coalgebras over a monad}}$$

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Saturation for LTS

Remark

Weak bisimulation on α is defined as strong bisimulation on a saturated structure α^*.

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Question

Can we describe saturation categorically?

The Kleisli category for the LTS monad $T = \mathcal{P}(\Sigma_\tau \times \mathcal{I}d)$ is order enriched with the hom-posets admitting arbitrary joins.

$$\alpha^* = 1 \vee \alpha \vee \alpha^2 \vee \ldots = \bigvee_{n=0}^{\infty} \alpha^n.$$

Reminder

The composition in $\mathcal{K}l(\mathcal{P}(\Sigma_\tau \times \mathcal{I}d))$ is given by:

$$g \cdot f(x) = \{(\sigma, z)) \mid x \xrightarrow{\sigma}_f y \xrightarrow{\tau}_g z \text{ or } x \xrightarrow{\tau}_f y \xrightarrow{\sigma}_g z\}.$$

Saturation for LTS

In other words, given an endomorphism $\alpha : X \to X$ in the Kleisli category for the LTS monad (a.k.a. a labelled transistion system) the endomorphism $\alpha^* : X \to X$ is the least arrow satisfying:

(a) $\alpha \leqslant \alpha^*$,

(b) $1 \leqslant \alpha^*$ and $\alpha^* \cdot \alpha^* \leqslant \alpha^*$.

Observation

We can think of α^* as the reflexive and transitive closure of α.

Saturation for LTS categorically

$$\mathrm{End}^{\leq}(\mathit{Kleisli}) \underset{\longleftarrow}{\overset{(-)^*}{\longrightarrow}} \mathrm{End}^{\leq *}(\mathit{Kleisli}), \qquad \perp$$

where:

- *Kleisli* = the Kleisli category for the LTS monad,
- $\mathrm{End}^{\leq}(\mathsf{K})$ = category of endomorphisms in an order enriched category K as objects. An arrow $f : X \to Y$ in K is a morphism between $\alpha : X \to X$ and $\beta : Y \to Y$ in $\mathrm{End}^{\leq}(\mathsf{K})$ whenever

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \alpha \downarrow & \leq & \downarrow \beta \\ X & \xrightarrow[f]{} & Y \end{array}$$

- $\mathrm{End}^{\leq *}(\mathsf{K})$ = the full subcategory of $\mathrm{End}^{\leq}(\mathsf{K})$ with endomorphisms satisfying $1 \leq \alpha$ and $\alpha \cdot \alpha \leq \alpha$ as objects.

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Saturation in general

Let K be an order enriched category. We say that K admits *saturation* if the inclusion functor admits a left adjoint:

$$\mathsf{End}^{\leq}(\mathsf{K}) \underset{}{\overset{(-)^*}{\rightleftarrows}} \perp \ \mathsf{End}^{\leq *}(\mathsf{K}).$$

Theorem

Assume that

- hom-posets of K admit binary joins,
- K is ωCPO-enriched,
- K satisfies *left distributivity*, i.e. $f \cdot (g \vee h) = f \cdot g \vee f \cdot h$

then K admits saturation.

Examples of systems whose Klesli cats admit saturation

- Labelled transition systems ($\mathcal{P}(\Sigma_\tau \times \mathcal{I}d)$-coalgebras),
- Segala systems (viewed as $\mathcal{CM}(\Sigma_\tau \times \mathcal{I}d)$-coalgebras),
- Topological Kripke frames,
- ...

Negative example

Fully probabilistic systems (even if viewed as $\mathbb{F}_{[0,\infty]}(\Sigma_\tau \times \mathcal{I}d)$-coalgebras instead of $\mathcal{D}(\Sigma_\tau \times \mathcal{I}d)$-coalgebras).

Fact

Although the Kleisli cat for the monad $\mathbb{F}_{[0,\infty]}(\Sigma_\tau \times \mathcal{I}d)$ admits binary joins and is ωCPO-enriched it is *not* left distributive.

What to do if there's no left distributivity...

If K is not left distributive but admits binary joins and is ωCPO-enriched then ...

What to do if there's no left distributivity...

If K is not left distributive but admits binary joins and is ωCPO-enriched then ... we embed K into

$$\widehat{\mathsf{K}} = Lax(\mathsf{K}, \omega\mathsf{CPO}^{\vee})^{op}_{oplax}$$

which is additionally left distributive. The embedding:

- preserves order,
- is *locally reflective*. I.e. for any two objects $X, Y \in \mathsf{K}$ the hom-poset restriction $\mathsf{K}(X, Y) \to \widehat{\mathsf{K}}(\widehat{X}, \widehat{Y})$ of the embedding admits a left adjoint $\Theta_{X,Y}$ (or simply Θ).

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Weak behavioural equivalence for cats with saturation

Let K be a category which admits saturation and let J be a subcategory of K with all objects from K (conf. $J = \text{Set}$ and $\mathsf{K} = \mathcal{K}l(\mathcal{P}(\Sigma_\tau \times \mathcal{I}d))$).

Definition

We say that a morphism $f : X \to Y \in J$ is a *weak behavioural morphism* on $\alpha : X \to X \in \mathsf{K}$ if there is $\beta : Y \to Y \in \mathsf{K}$ making:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \alpha^*}\downarrow & & \downarrow{\scriptstyle \beta} \\ X & \xrightarrow[f]{} & Y \end{array}$$

In other words, $f \circ \alpha^* = \beta \circ f$. *Weak beh equivalence* on α is the kernel of a weak behavioural morphism on α.

Weak behavioural morphisms in general

Definition

Given an endomorphism $\alpha : X \to X$ in K a morphism $f : X \to Y \in J$ is *weak behavioural morphism* provided that there is $\beta : Y \to Y$ such that

$$\Theta(\widehat{f} \circ \widehat{\alpha}^*) = \Theta(\widehat{\beta} \circ \widehat{f})$$

$$\begin{array}{ccccc} X & & \widehat{X} & & \widehat{X} \\ \downarrow{\scriptstyle \alpha} & \mapsto & \downarrow{\scriptstyle \widehat{\alpha}} & \mapsto & \downarrow{\scriptstyle \widehat{\alpha}^*} \\ X & & \widehat{X} & & \widehat{X} \end{array}$$

and come back...

Weak behavioural morphisms in general

Theorem

A morphism f in J is a weak behavioural morphism on α provided that there is β satisfying:

$$\alpha_f^* = \beta \circ f,$$

where $\alpha_f^* = \mu x.(f \vee x \circ \alpha)$.

Interesting remarks

Observation

The definition of weak behavioural morphism does not depend on the choice of $\widehat{\mathsf{K}}$ we embed K into.

This explains why if K is left distributive then we can take $\widehat{\mathsf{K}} = \mathsf{K}$. In this case the identity embedding is locally reflective. Hence,

$$\underline{\Theta(\widehat{f} \circ \widehat{\alpha}^*) = \Theta(\widehat{\beta} \circ \widehat{f})}$$

Interesting remarks

Observation

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$$\frac{\Theta(\widehat{f} \circ \widehat{\alpha}^*) = \Theta(\widehat{\beta} \circ \widehat{f})}{f \circ \alpha^* = \beta \circ f}$$

Systems that fall into our framework

- Labelled transition systems,
- Segala systems (viewed as $\mathcal{CM}(\Sigma_\tau \times \mathcal{I}d)$-coalgebras),
- Fully probabilistic systems (viewed as $\mathbb{F}(\Sigma_\tau \times \mathcal{I}d)$-coalgebras),
- Topological Kripke frames,
- Continuous-state stochastic systems
- ...

Future work

- Lax functors, e.g. $Lax(\mathbb{N}, \mathsf{K}) \rightleftarrows Lax(1, \mathsf{K})$,
- timed automata,
- ...

Bilbiography

1. T. Brengos, *Weak bisimulations for coalgebras over ordered monads*, LMCS 11 (2:14), 2015
2. T. Brengos, M. Miculan, M. Peressotti *Behavioural equivalences for coalgebras with unobservable moves*, JLAMP 67, 2015