Behavioural equivalences for coalgebras with unobservable moves

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Abstract. We introduce a general categorical framework for the definition of weak behavioural equivalences, building on and extending recent results in the field. This framework is based on special *order enriched categories*, i.e. categories whose hom-sets are endowed with suitable complete orders. Using this structure we provide an abstract notion of *saturation*, which allows us to define various (weak) behavioural equivalences. We show that the Kleisli categories of many common monads are categories of this kind. On one hand, this allows us to instantiate the abstract definitions to a wide range of existing systems (weighted LTS, Segala systems, calculi with names, etc.), recovering the corresponding notions of weak behavioural equivalences; on the other, we can readily provide new weak behavioural equivalences for more complex behaviours, like those definable on presheaves, topological spaces, measurable spaces, etc.

The notion of strong bisimulation for different transition systems plays an important role in theoretical computer science and has been well captured coalgebraically. However, when we come to behavioural equivalences for systems with unobservable (i.e., internal) moves, the situation is not as clear. The point is that what is "unobservable" depends on the system: in LTSs these are internal steps (the so-called "τ-transitions"), but in systems with quantitative aspects or dealing with resources internal steps may still have observable effects. This has led to many definitions, often quite *ad hoc*.

This situation points out the need for a general, uniform framework covering many weak behavioural equivalences at once. Here, we propose to host these constructions in *order-enriched categories* whose hom-sets are additionally endowed with *binary joins* for "merging" approximants, and a *complete order* to guarantee convergence of approximant chains. In this setting we can define, and solve, the abstract equations corresponding to many kinds of weak observational equivalence. For example, we will show the abstract schemata corresponding to Milner's and Baier-Hermann's versions of weak bisimulations; hence, these two different bisimulations are applications of the same general framework. Then, we show that the Kleisli categories of many monads commonly used for defining behavioural functors meet these mild requirements; this allows us to port the definitions above to a wide range of behaviours in different categories (such as presheaf categories, topological spaces and measurable spaces).

Following [1] the coalgebras with internal moves are modelled as coalgebras $X \to TX$ for a monad T on a category C. Hence, T-coaglebras are endomorphisms in the Kleisli category $\mathcal{K}l(T)$. The category C, which is a subcategory of $\mathcal{K}l(T)$ with all objects from it forms the source of behavioural morphisms. This allows us to abstract away from the specific type of coalgebras and assume we are given a $\omega\mathsf{CPO}^{\vee}$-enriched[3] category K and a subcategory J of K with all objects of K, generalizing the C and $\mathcal{K}l(T)$ scenario.

A *saturated endomorphism* $\alpha : X \to X$ in the category K is an endomorphism which satisfies $id \leq \alpha$ and $\alpha \circ \alpha \leq \alpha$. Intuitively, these two inequalities can be understood as reflexivity and transitivity of α respectively. The main ingredient of the definition of a weak behavioural morphism is the so-called *saturation*, i.e. a reflexive and transitive closure of endomorphisms (we denote by $(-)^*$) which additionally preserves weak homomorphisms.

Theorem 1. *If* K *is additionally left distributive then it admits saturation.*

However, although all examples we consider in [2] form Kleisli categories which are $\omega\mathsf{CPO}^{\vee}$-enriched not all are left distributive. In order to achieve left distributivity (and, hence, saturation admittance) we embed K into the the dual category to the category of all lax functors $\mathsf{K} \to \omega\mathsf{CPO}^{\vee}$ and oplax transformations, we denote by $\widehat{\mathsf{K}}$.

Theorem 2. *The category* $\widehat{\mathsf{K}}$ *is left distributive and* $\omega\mathsf{CPO}^{\vee}$*-enriched and hence admits saturation.*

Although, the embedding $\widehat{(-)} : \mathsf{K} \to \widehat{\mathsf{K}}$ allows us to achieve saturation, we have to be able go back to the underlying category K in order to define weak behavioural morphisms for its endomorphisms. Remarkably, this is done by realizing that the hom-poset restriction $\widehat{(-)}_{X,Y} : \mathsf{K}(X,Y) \to \widehat{\mathsf{K}}(\widehat{X},\widehat{Y})$ of the embedding admits a left adjoint Θ. We say that an arrow $f : X \to Y$ in J is *weak behavioural morphism* on $\alpha : X \to X \in \mathsf{K}$ provided that there is an endomorphism $\beta : Y \to Y \in \mathsf{K}$ such that: $\Theta(\widehat{f} \circ \widehat{\alpha}^*) = \Theta(\widehat{\beta} \circ \widehat{f})$.

This construction readily and precisely instantiates to known notions of weak behavioural equivalences for systems like e.g., LTS, fully-probabilistic systems, Segala systems, weighted LTS, the π-calculus and new ones as well such e.g. Vietoris and measurable systems (defined in KHaus and Meas, respectively). We refer the interested reader to [2] for further details and examples.

References

1. Tomasz Brengos. Weak bisimulation for coalgebras over order enriched monads. *Logical Methods in Computer Science*, 11(2:14):1–44, 2015.
2. Tomasz Brengos, Marino Miculan, and Marco Peressotti. Behavioural equivalences for coalgebras with unobservable moves. *Journal of Logical and Algebraic Methods in Programming*, 84(6):826–852, 2015.

[3] An $\omega\mathsf{CPO}^{\vee}$-enriched category is an order enriched category which satisfies the following two properties: (1) any ω-ascending chain of morphisms admits a supremum which is preserved by the composition; (2) any pair of morphisms with common domain and codomain admits a join (which is *not* necessarily preserved by the composition).