Affine Monads and Side-Effect-Freeness

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Outline

Where we are, sofar

Context: effectus theory and side-effects

Affine monads

Predicates and instruments

Main results

Conclusions

Quantum

Quantum

- Quantum computation and logic is a fascinating area
- "Hot topic", because of all the buzz about quantum computers
- Potentially large impact, esp. in security
  - existing (public key) algorithms are vulnerable
  - new research area: "post-quantum crypto"
- New challenges for existing concepts in (theoretical) CS
  - three overlapping areas: physics, math, CS
  - John Baez: category theory is "Rosetta Stone"
- Strong coalgebraic flavour
  - "states" play an important role
  - quantum observations can have a side-effect (state-change)

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- If predicates can have side-effects, commutativity is no longer obvious. Conjunction should be used as 'and-then'
- This plays an important role in the quantum world — and also in imperative programming where & (and &&) are not commutative

Effectus theory

Effectus theory

- Own (group's) work has led to a new categorical notion: effectus
  - it's a certain kind of category, with $0, +, 1$, some pullbacks, and some jointly monic maps
  - its predicates form effect modules, its states are convex sets, and together they form a "state-and-effect" triangle

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- **We have**: in the probabilistic and Boolean case, instruments are side-effect-free, but not in the quantum case!

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general
'non-commutative'
effectuses

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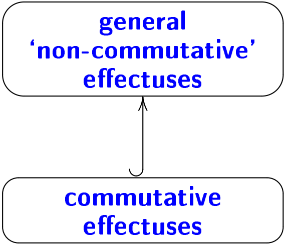

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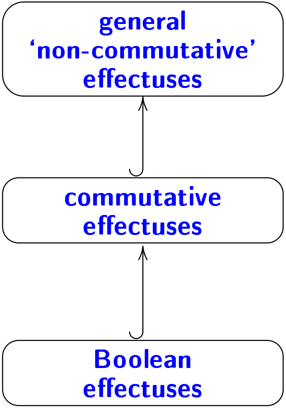

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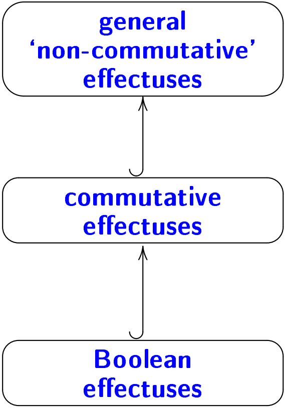

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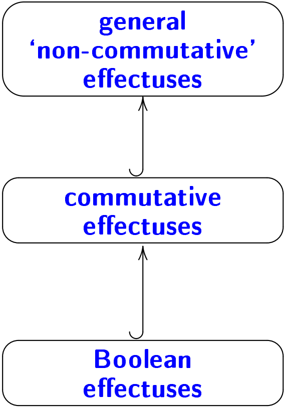

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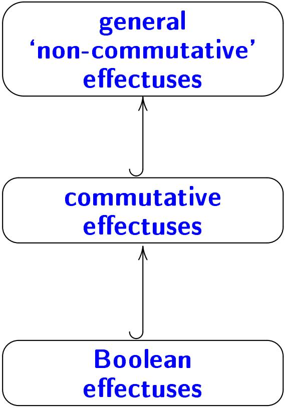

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Theorem (See Effectus Intro paper on ArXiv)

The Boolean effectuses are precisely the extensive categories (with 1*).*

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Examples: $\mathcal{K}\ell(\mathcal{D}) \quad \mathcal{K}\ell(\mathcal{G}) \quad \mathcal{K}\ell(\mathcal{E}) \quad \mathcal{K}\ell(\mathcal{R}) \simeq \mathbf{CvNA}^{\mathrm{op}} \quad \ldots$

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Main question underlying the CMCS paper

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These questions have "good" answers

- they are first steps towards the *wild conjecture*

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- The monad is strong if there is a strength map $\mathrm{st}_1\colon T(X) \times Y \to T(X \times Y)$ suitably commuting with other structure
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$$\begin{array}{ccccccc}
 & \overset{\mathrm{st}_1}{\nearrow} & T(X \times T(Y)) & \xrightarrow{T(\mathrm{st}_2)} & T^2(X \times Y) & \overset{\mu}{\searrow} & \\
T(X) \times T(Y) & & & & & & T(X \times Y) \\
 & \underset{\mathrm{st}_2}{\searrow} & T(T(X) \times Y) & \xrightarrow[T(\mathrm{st}_1)]{} & T^2(X \times Y) & \underset{\mu}{\nearrow} &
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Note: if T is affine, then 1 is final in $\mathcal{K}\ell(T)$.

Affine submonad

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Assuming enough pullbacks, the affine submonad $T_a \rightarrowtail T$ is defined via:

$$\begin{array}{ccc} T_a(X) & \longrightarrow & 1 \\ \big\downarrow & \lrcorner & \big\downarrow \eta \\ T(X) & \xrightarrow[T(!)]{} & T(1) \end{array}$$

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- *if T is strong / commutative then so is T_a*

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$$\mathcal{P}_a(X)$$

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Example: maps $X \to \mathcal{D}(Y)$ are causal maps $X \to \mathcal{M}_{\mathbb{R}_{\geq 0}}(Y)$.

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- In many (probabilistic) examples, predicates are maps $X \to [0,1]$

Instruments

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For a predicate $p\colon X \to 1+1$ we define an instrument $\mathrm{instr}_p\colon X \to X + X$ in $\mathcal{K}\ell(T)$ as:

$$\mathrm{instr}_p = \Big(X \xrightarrow{\langle p, \mathrm{id}\rangle} T(2) \times X \xrightarrow{\mathrm{st}_1} T(2 \times X) \xrightarrow{\cong} T(X + X)\Big)$$

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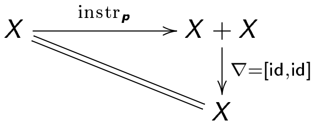

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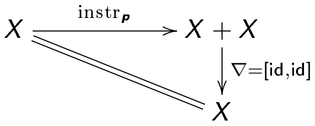

Lemma

If T is affine, then each instrument is side-effect-free

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- For the (affine) non-empty powerset the case $p(x) = \emptyset$ does not occur, so we get side-effect-freeness.

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This instrument incorporates the side-effects of the predicate p

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\end{aligned}$$

Where we are, sofar

Strong affineness

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Definition

A (strong) monad T is called strongly affine if the following squares are pullbacks

$$\begin{array}{ccc} T(X) \times Y & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \mathrm{st}_1}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ T(X \times Y) & \xrightarrow[T(\pi_2)]{} & T(Y) \end{array}$$

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(Strongly affine implies affine)

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- (Kenta Cho) The monad $\mathcal{D}_\pm$ is affine but not strongly affine
  - $\mathcal{D}_\pm(X)$ contains $\sum_i r_i | x_i \rangle$ with $r_i \in \mathbb{R}$ and $\sum_i r_i = 1$
  - In this monad $\mathcal{D}_\pm$ there is interference: positive and negative factors can cancel each other out

Strongly affine monads and instruments

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If T is strongly affine, then there is a bijective corrrespondence

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$$\frac{\textit{predicates}}{\overline{\overline{\textit{side-effect-free instruments}}}}$$

More precisely, the correspondence is between maps in $\mathcal{K}\ell(T)$,

$$\frac{X \xrightarrow{\;p\;} 2}{\overline{\overline{X \xrightarrow[f]{} X + X \quad \textit{with } \nabla \bullet f = id}}}$$

Relating commutativity

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Theorem (II)

If the monad T is commutative, then instruments commute — giving commutativity in an effectus-theoretic sense.

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More precisely, for predicates $p, q \colon X \to 2$ we have:

$$\begin{array}{ccccc} X & \xrightarrow{instr_{p}} & X+X & \xrightarrow{q+q} & 2+2 \\ \| & & & & \cong \big\downarrow [\kappa_1+\kappa_1, \kappa_2+\kappa_2] \\ X & \xrightarrow[instr_{q}]{} & X+X & \xrightarrow[p+p]{} & 2+2 \end{array}$$

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X & \xrightarrow[\;instr_q\;]{} & X + X & \xrightarrow[\;p+p\;]{} & 2 + 2
\end{array}
$$

The isomorphism on the right can be illustrated as:

$$
\begin{array}{ccccccccccc}
2+2 & = & (1 & + & 1) & + & (1 & + & 1) \\
 & & \downarrow & & \searrow & & \swarrow & & \downarrow \\
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$$

Where we are, sofar

Final remarks

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- Quantum theory forms a rich source of inspiration for program semantics and logic — and for coalgebra in particular
- Recent formalisation in terms of effectuses
    - framework deals with side-effects of observations
    - Boolean and probabilitistic computation given by subclasses
- Characterising the commutative (probabilistic and side-effect-free) fragment is an open challenge
    - Kleisli categories of suitable monads play an important role
- This CMCS paper clarifies the role of strong affiness and of commutativity of the monad