Regular Behaviours with Names

On Rational Fixpoints of Endofunctors on Nominal Sets*

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Behaviours with Names. Nominal sets constitute a conveniently abstract formalism to talk about names, renaming, freshness, α-equivalence, and binding of names. Concretely, for a fixed set of names $\mathcal{V}$, a nominal set X *(a)* is an $\mathfrak{S}_{\mathsf{f}}(\mathcal{V})$-set, where $\mathfrak{S}_{\mathsf{f}}(\mathcal{V})$ is the group of finite permutations on $\mathcal{V}$ and *(b)* only has finitely supported elements, i.e. for each $x \in X$, there is a least finite set $\mathsf{supp}(x)$, s.t. any $\pi \in \mathfrak{S}_{\mathsf{f}}(\mathcal{V})$ that fixes every element in $\mathsf{supp}(x)$ also fixes x. Intuitively, $\mathsf{supp}(x)$ is the set of names occurring free in x.

Nominal sets together with *equivariant* maps – maps that preserve the group action – form the category Nom. Coalgebras for endofunctors on Nom model, e.g., various flavours of automata, and (possibly) infinite terms involving variable binding. Many Nom-functors of interest arise from a common pattern: they are either liftings of Set-functors or quotients of such liftings. For example, λ-terms can be represented as the initial algebra for one of the functors

$$LX = \mathcal{V} + X \times X + \mathcal{V} \times X \quad \xrightarrow{[-]_\alpha} \quad L_\alpha X = \mathcal{V} + X \times X + [\mathcal{V}]X$$

where L corresponds to raw λ-terms and its natural quotient L_α to λ-terms modulo α-equivalence. Because of the restriction to finite support, the respective final coalgebras are $\nu L = \lambda$-trees involving finitely many variables, and $\nu L_\alpha = \lambda$-trees with finitely many free but possibly infinitely many bound variables [1]. This illustrates two ways in which Nom differs from Set: although L is a lifting of a Set functor, νL is different in Nom than in Set, and although L_α is a quotient of L, νL_α is *not* a quotient of νL.

Regular Behaviours. While the final coalgebra contains *all* possible behaviours, one is often only interested in the subclass of *regular* behaviours; that is, behaviours with a finite description. Categorically, this notion is captured by the *rational fixpoint* of a finitary endofunctor F on an lfp-category. In Set, the rational fixpoint ϱF is the subcoalgebra of the final coalgebra given by the union of images of all coalgebras with a finite carrier. Concrete instances are: for $FX = 2 \times X^A$ (with A finite), ϱF is the class of regular languages over A; for a signature functor F_Σ, ϱF_Σ is the class of rational Σ-trees, i.e. trees with only finitely many subtrees (up to isomorphism).

The Rational Fixpoint in Nom. Similarly, the rational fixpoint of a Nom-functor is the subcoalgebra of the final coalgebra given by the union of images of all *orbit-finite* coalgebras (orbit-finite means having finitely many elements up to renaming). However, using this to derive a concrete description of the rational fixpoint can be non-trivial, see e.g. the proof that ϱL_α is the set of rational λ-trees [2]; in particular, one first needs a concrete description of the final coalgebra.

* Full version at `http://www8.cs.fau.de/ext/thorsten/nomliftings.pdf`

In the following, we will present sufficient conditions respectively ensuring that *(a)* for a lifted functor, the rational fixpoint lifts from Set to Nom, and that *(b)* for a quotient of a functor, the rational fixpoint is just the level-wise quotient of the former.

Rational Fixpoints of Localizable Liftings. In the following, we consider a finitary Nom-functor that is lifted from Set, and, for convenience, preserves monomorphisms.

Definition 1. A Nom-functor $\bar{F}$ is a *localizable lifting* of a Set-functor F if *(a)* $U\bar{F} = FU$, where U denotes the forgetful functor $\mathsf{Nom} \to \mathsf{Set}$, and *(b)* $\bar{F}$ is induced by a distributive law $\lambda : (\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times -)F \to F(\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times -)$ that can be restricted to any subset $W \subseteq \mathcal{V}$.

The class of mono-preserving, finitary, and localizable liftings is closed under finite products, arbitrary coproducts, and functor composition, and contains the identity functor and all constant functors; in particular, polynomial functors like L belong to this class. Moreover, any finitary Set-functor induces such a localizable lifting.

Theorem 2. *Let $\bar{F}$ be a localizable lifting of F. Then the rational fixpoint ϱF, equipped with a nominal structure defined by corecursion, is the rational fixpoint of $\bar{F}$.*

Rational Fixpoints of Quotients. Fix a finitary functor G on Nom and a quotient $q : G \twoheadrightarrow H$.

Definition 3. For nominal sets X, Y, write $X < Y = \{(x, y) \mid \mathsf{supp}(x) \subseteq \mathsf{supp}(y)\} \subseteq X \times Y$. A *sub-strength* of G is a family of equivariants $s_{X,Y} : GX < Y \to G(X < Y)$ that commutes with the left projection: $\mathsf{pr}_1 = F\,\mathsf{pr}_1 \cdot s_{X,Y}$.

This condition is rather weak; the identity and any constant Nom-functor have a sub-strength, and having a sub-strength is closed under finite products, arbitrary coproducts and functor composition.

Theorem 4. *If G has a substrength, then the rational fixpoint $(\varrho H, h)$ is a quotient of $(\varrho G, g)$. Specifically, the unique H-coalgebra homomorphism $(\varrho G, q_{\varrho G} \cdot g) \to (\varrho H, h)$ is an epimorphism.*

Examples. Theorems 2 and 4 imply that if a Nom-functor H is a quotient of the lifting of a polynomial Set-functor F, then ϱH is just the level-wise quotient of the Set-coalgebra ϱF. This pattern applies to various functors:

(a) For any binding signature Σ (see [1]), the rational fixpoint contains the rational Σ-trees modulo α-equivalence, in particular for the signature of λ-trees ($F = L$, $H = L_\alpha$).

(b) Exponentiation $HX = X^P$ in Nom by an orbit-finite P is the quotient of $FX = X \times \mathcal{P}_\mathsf{f}(\mathcal{V}) \times \coprod_{n\in\mathbb{N}} P^n \times X^n$. So we obtain descriptions of the rational fixpoints for functors used for nominal automata, e.g. $2 \times (-)^A$, A orbit-finite, $2 \times (-)^\mathcal{V} \times [\mathcal{V}](-)$, or $2 \times \mathcal{P}_\mathsf{f}((-)^\mathcal{V}) \times \mathcal{P}_\mathsf{f}([\mathcal{V}](-))$.

References

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2. Milius, S., Wißmann, T.: Finitary corecursion for the infinitary lambda calculus. In: Algebraic and Coalgebraic Methods in Computer Science, CALCO 2015. LIPiCS (2015)