Towards Coalgebraic Semantics of Higher-Order Behaviours

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April 3, 2016

Motivation

> *For **first-order** languages, there is a **common consensus about** what **bisimulation** is and how it should be defined, and the associated proof techniques are well-developed [...]. The picture is **less clear for higher-order** languages, as available definitions and proof techniques are often difficult to adapt to different languages.*

D. Sangiorgi, N. Kobayashi, E. Sumii, *Environmental bisimulations for higher-order languages*

What is an higher-order behaviour?

> *Higher-order is… an insidious beast!*
>
> *Davide Sangiorgi*

What is an higher-order behaviour?

π-calculus, Ambient calculus, any language with `eval`

- × dynamic process topologies
- × code mobility
- × encodings of λ etc. do not reflect bisimilarity.

λ-calculus, HO-Core, HO-π-calculus, …

- ✓ values can be terms/processes/automata
- ✓ systems operate on systems of *the same kind*
- ✓ *all* systems are a values

Higher-order via term/process passing

Consider a value passing behaviour, e.g.:

$$X \to X^V + V$$

where V is the set of values.

if $x \sim y$ then

- $x \xrightarrow{v} x'$ then $y \xrightarrow{v} y'$ and $x' \sim y'$;
- $x \to v$ then $y \to v$;
- the symmetric.

Assume processes are exchanged ($V \cong Proc$)

$$x \sim y, x \xrightarrow{p} x', y \xrightarrow{q} y', p \sim q \;\not\Rightarrow\; x' \sim y'$$

i.e. we can (craft contexts that) disinguish bisimilar processes

Higher-order via term/process passing

Assume $V = \mathit{Proc}$ and $\approx \subseteq V \times V$

if $x \sim y$ then

- $x \xrightarrow{p} x'$ then for any $p \approx q$ $y \xrightarrow{q} y'$ and $x' \sim y'$;
- $x \to p$ then $y \to q$ and $p \approx q$;
- the symmetric conditions.

$\approx$ has to satisfy the above conditions w.r.t process dynamics ($\sigma\colon V \to V^V + V$)

Intuitively, we need processes and their dynamics to be

- strongly extensional (cf. $V/\approx$);
- sound w.r.t. final semantics ($\approx$ is a bisimulation)
- complete w.r.t. final semantics (every behaviour is a value)

Reworded...

$\sigma\colon V \to V^V + V$ should be $\nu(\mathit{Id}^V + V)$

Higher-order behaviours via behaviour passing

Abstract endofunctors modelling value passing behaviours as:

$$F\colon \mathsf{C}^{op} \times \mathsf{C} \to [\mathsf{C}, \mathsf{C}]$$

We say that $B\colon \mathsf{C} \to \mathsf{C}$ *belongs to* F iff $B \cong F(V, W)$ for some $V, W \in \mathsf{C}$.

Definition (Higher-order behaviours)

$B\colon \mathsf{C} \to \mathsf{C}$ models *higher-order behaviours* in $F\colon \mathsf{C}^{op} \times \mathsf{C} \to [\mathsf{C}, \mathsf{C}]$ iff

$$B \cong F(|\nu B|, |\nu B|). \qquad (\circlearrowleft)$$

Solving (↻)

Theorem

Assume C Cpo-*algebraically compact*[1]. *For any* Cpo-*enriched*

$$F\colon \mathrm{C}^{op} \times \mathrm{C} \to [\mathrm{C}, \mathrm{C}]$$

there exists a unique (up-to iso) $B\colon \mathrm{C} \to \mathrm{C}$ *such that:*

- *B is a solution to* (↻):
$$B \cong F(|\nu B|, |\nu B|)$$
- *B dominates any other solution to* (↻):
$$B' \cong F(|\nu B'|, |\nu B'|) \implies B \lhd B'$$

[1] Any Cpo-functor on C has both an initial algebra and a final coalgebra and these are canonically isomorphic.

Solving (↺)

Proof (sketch):

Lemma: $|\nu - |\colon [\mathrm{C}, \mathrm{C}] \to \mathrm{C}$.

$$\begin{cases} X \cong F(|\nu Y|, |\nu X|) \\ Y \cong F^{op}(|\nu X|, |\nu Y|) \end{cases} \quad \mapsto \quad \begin{array}{l} G\colon [\mathrm{C}, \mathrm{C}]^{op} \times [\mathrm{C}, \mathrm{C}] \to [\mathrm{C}, \mathrm{C}]^{op} \times [\mathrm{C}, \mathrm{C}] \\ G \triangleq \langle F(|\nu - |, |\nu - |), F^{op}(|\nu Y|, |\nu X|)\rangle \end{array}$$

Open problem: if C is algebraically compact then so is $[C, C]$.

Lemma: if C algebraically compact then $\mu G \cong \nu G$.

The claim follows by standard (categorical) domain theory machinery.

Solving $(\circlearrowleft)$

$$
\begin{array}{ccccccccccc}
1 & \longleftarrow & 1 & \longleftarrow & 1 & \longleftarrow & \cdots & & 1 & & \\
\twoheaduparrow\ \downarrowtail & & \twoheaduparrow\ \downarrowtail & & \twoheaduparrow\ \downarrowtail & & & & \twoheaduparrow\ \downarrowtail & & \\
1 & \longleftarrow & F_{1,1}(1) & \longleftarrow & F^2_{1,1}(1) & \longleftarrow & \cdots & & Z_1 & \cong & F_{1,1}(Z_1) \\
\twoheaduparrow\ \downarrowtail & & \twoheaduparrow\ \downarrowtail & & \twoheaduparrow\ \downarrowtail & & & & \twoheaduparrow\ \downarrowtail & & \\
1 & \longleftarrow & F_{Z_1,Z_1}(1) & \longleftarrow & F^2_{Z_1,Z_1}(1) & \longleftarrow & \cdots & & Z_2 & \cong & F_{Z_1,Z_1}(Z_2) \\
\twoheaduparrow\ \downarrowtail & & \twoheaduparrow\ \downarrowtail & & \twoheaduparrow\ \downarrowtail & & & & \twoheaduparrow\ \downarrowtail & & \\
1 & \longleftarrow & F_{Z_2,Z_2}(1) & \longleftarrow & F^2_{Z_2,Z_2}(1) & \longleftarrow & \cdots & & Z_3 & \cong & F_{Z_2,Z_2}(Z_3) \\
\twoheaduparrow\ \downarrowtail & & \twoheaduparrow\ \downarrowtail & & \twoheaduparrow\ \downarrowtail & & & & \twoheaduparrow & & \\
\vdots & & \vdots & & \vdots & & \ddots & & \vdots & & \\
1 & \longleftarrow & F_{Z,Z}(1) & \longleftarrow & F^2_{Z,Z}(1) & \longleftarrow & \cdots & & Z & \cong & F_{Z,Z}(Z)
\end{array}
$$

Finite order approximations

$$\frac{\text{higher-order} = \lim_{n\to\omega} n\text{-finite-order}}{\text{amalgamation in } \mathrm{Sh}_{\mathrm{C}}(\omega)}$$

Proofs in terms of finite-order approximations:

$$x \sim y \iff \forall n < \omega \; x|_n \sim_n y|_n$$

- Proofs by well-founded induction on finite-order
- Internal language of $\mathrm{Sh}_{\mathrm{C}}(\omega)$ (*cf.*, topos of trees)
- Techniques developed for step-indexed relations

Conclusion and future work

Done:

- HO behaviours via behaviour passing (↺)
- Solving (↺)
  - $F\colon \mathsf{D}^{op} \times \mathsf{D} \to [\mathsf{C}, \mathsf{C}]$ (where D is alg.comp. but C is not)
  - $\mathrm{Sh}_{\mathsf{C}}(\omega)$ (*cf.*, proofs by induction on finite-order approximations)
- Abstract SOS (with binders)
  - Notion of soundness and completeness of term passing SOS specifications w.r.t behaviour passing ones.

To-do list:

- Environmental bisimulation (*et similia*)
- Saturation & up-to

Thanks for your attention