An Eilenberg–like theorem for algebras over a monad

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In the classical Eilenberg theorem there are two key notions: pseudovarieties of monoids and pseudovarieties of recognizable languages. A *pseudovariety of monoids* is a class of finite monoids that is closed under homomorphic images, submonoids, and finite products. A *pseudovariety of recognizable languages* is an operator $\mathcal{L}$ such that $\mathcal{L}\Sigma \subseteq 2^{\Sigma^*}$ for every finite alphabet Σ and it satisfies the following two properties:

i) $\mathcal{L}\Sigma$ is a set of recognizable languages over Σ and is a Boolean algebra closed under left and right derivatives.
ii) $\mathcal{L}$ is closed under morphic preimages: for every finite alphabet Γ, a language $L \in \mathcal{L}\Sigma$, and a homomorphism of monoids $h : \Gamma^* \to \Sigma^*$, we have that $L \circ h \in \mathcal{L}\Gamma$.

The classical Eilenberg theorem states that there is a one–to–one correspondence between pseudovarieties of monoids and pseudovarieties of recognizable languages [3, Theorem 3.4]. There is also a slightly different version, [3, Theorem 3.4s], for which the algebras considered are semigroups instead of monoids.

Some Eilenberg–like theorems are known in the literature (e.g. [4–6]) in which other algebraic structures are considered than monoids, and the Boolean algebra condition in i) is weakened. For example, in the categorical framework setup in [1, 2] new proofs of the classical Eilenberg theorem have been found and many new Eilenberg–like theorems have been (re)discovered.

The main idea of this work is to combine the ideas from [2] and [7], by using the setting of lifting dualities to Eilenberg–Moore categories given in [7], from which one could be able to recover the Eilenberg–like theorems included in [1, 2] and derive new ones. For this purpose, we can start with a duality $\mathcal{C}^{op} \cong \mathcal{D}$ and a monad T on $\mathcal{D}$. Now, by using [7, Proposition 14] we get a canonical comonad B on $\mathcal{C}$ such that the duality $\mathcal{C}^{op} \cong \mathcal{D}$ is lifted to their corresponding Eilenberg–Moore categories to get a duality $\mathrm{Coalg}(\mathsf{B})^{op} \cong \mathrm{Alg}(\mathsf{T})$. From this setting we define *pseudovarieties of* T*–algebras* as classes of finite T–algebras that are closed under homomorphic images, subalgebras, and finite products, as it is done in [2]. This approach eliminates the main restriction on the kind of algebras considered in [1]. On the other hand, *pseudovarieties of* T*–languages* are taken from the category $\mathcal{C}$. This eliminates the restriction we have in [2] that every $\mathcal{L}\Sigma$ is a Boolean algebra by allowing this definition to depend on the category $\mathcal{C}$ we consider.

In [1] the main setting is to consider two categories $\mathcal{C}$ and $\mathcal{D}$ that are *predual*, i.e. they form a duality if we restrict them to finite objects. In this case pseudovarieties of languages are taken from $\mathcal{C}$ and pseudovarieties of algebras

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are taken from $\mathcal{D}$ by considering the notion of a $\mathcal{D}$–monoid. With this approach they derive some of the well–known Eilenberg–like theorems such as [3, Theorem 3.4] and [4, Theorem 5.8], but all the algebras considered there have a monoid structure because of this $\mathcal{D}$–monoid approach (e.g. monoids, ordered monoids, idempotent semirings, $\mathbb{Z}_2$–algebras, and monoids with 0). This can be seen as a limitation on the kind of algebras considered (e.g. they cannot derive the semigroup version [3, Theorem 3.4s] by using this approach or versions for other algebraic structures).

In [2] the main setting is to consider a monad $\mathsf{T} = (T, \eta, \mu)$ to generalize the kind of algebras we can consider (e.g. the monad $TA = A^*$ for monoids, and the monad $TA = A^+$ for semigroups). In this case, pseudovarieties of algebras are classes of finite T-algebras closed under homomorphic images, subalgebras, and finite products. To define pseudovarieties of languages he restricts to the classical Eilenberg theorem by considering only Boolean algebras (he also considers two different kinds of derivatives: syntactic derivatives and polynomial derivatives. In some special cases those notions of derivatives coincide). This definition of pseudovarieties of languages can be seen as a limitation because for every pseudovariety of languages $\mathcal{L}$ the set $\mathcal{L}\Sigma$ has to be a Boolean algebra (we cannot derive Eilenberg-like theorems such as the ones in [4–6] from this setting).

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