An Eilenberg–like Theorem for Algebras on a Monad

Julian Salamanca (CWI Amsterdam)

April 2, 2016

CMCS 2016

Short contributions

Outline

- Classical Eilenberg theorem
  - Languages over an alphabet.
  - Finitely recognizable languages.
  - Pseudovarieties of monoids.
  - (Pseudo)varieties of finitely recognizable languages.
- Generalization for algebras on a monad T.
  - T–languages over an object.
  - Finitely recognizable T–languages.
  - Pseudovarieties of T–algebras.
  - Pseudovarieties of finitely recognizable T–languages.
  - Syntactic T–algebras.
- Conclusions.

Classical Eilenberg theorem

Given a set (alphabet) Σ, denote by Σ^* the free monoid on Σ.

Classical Eilenberg theorem

Given a set (alphabet) Σ, denote by Σ^* the free monoid on Σ.

A *language* over Σ is a subset L of Σ^*.

Classical Eilenberg theorem

Given a set (alphabet) Σ, denote by Σ^* the free monoid on Σ.

A *language* over Σ is a subset L of Σ^*.

L is *recognized* by a monoid A if there exists a monoid homomorphism $h : \Sigma^* \to A$ and $S \subseteq A$ s.t. $L = h^{-1}(S)$.

Classical Eilenberg theorem

Given a set (alphabet) Σ, denote by Σ^* the free monoid on Σ.

A *language* over Σ is a subset L of Σ^*.

L is *recognized* by a monoid A if there exists a monoid homomorphism $h : \Sigma^* \to A$ and $S \subseteq A$ s.t. $L = h^{-1}(S)$. L is *finitely recognizable* if it is recognized by a finite monoid.

Classical Eilenberg theorem

Given a set (alphabet) Σ, denote by Σ^* the free monoid on Σ.

A *language* over Σ is a subset L of Σ^*.

L is *recognized* by a monoid A if there exists a monoid homomorphism $h : \Sigma^* \to A$ and $S \subseteq A$ s.t. $L = h^{-1}(S)$. L is *finitely recognizable* if it is recognized by a finite monoid.

Given $a \in \Sigma$, the *right derivative* of L w.r.t. a is the language $L_a = \{w \in \Sigma^* \mid aw \in L\}$.

Classical Eilenberg theorem

Given a set (alphabet) Σ, denote by Σ^* the free monoid on Σ.

A *language* over Σ is a subset L of Σ^*.

L is *recognized* by a monoid A if there exists a monoid homomorphism $h : \Sigma^* \to A$ and $S \subseteq A$ s.t. $L = h^{-1}(S)$. L is *finitely recognizable* if it is recognized by a finite monoid.

Given $a \in \Sigma$, the *right derivative* of L w.r.t. a is the language $L_a = \{w \in \Sigma^* \mid aw \in L\}$.

The *left derivative* of L w.r.t. a is the language ${}_aL = \{w \in \Sigma^* \mid wa \in L\}$.

Pseudovarieties of monoids

Classical Eilenberg theorem

Pseudovarieties of monoids

Classes of finite monoids closed under homomorphic images (H), submonoids (S), and finite products (P_f).

Classical Eilenberg theorem

Pseudovarieties of monoids

Classes of finite monoids closed under homomorphic images (H), submonoids (S), and finite products (P_f).

(Pseudo)varieties of finitely recognizable languages

Classical Eilenberg theorem

Pseudovarieties of monoids

(Pseudo)varieties of finitely recognizable languages

Classes of finite monoids closed under homomorphic images (H), submonoids (S), and finite products (P_f).

Operators $\mathcal{L}$ s.t. $\forall\ \Sigma$ and Γ finite:

Classical Eilenberg theorem

Pseudovarieties of monoids

Classes of finite monoids closed under homomorphic images (H), submonoids (S), and finite products (P_f).

(Pseudo)varieties of finitely recognizable languages

Operators $\mathcal{L}$ s.t. $\forall\ \Sigma$ and Γ finite:

$\mathcal{L}\Sigma$ is a set of f.r.l. on Σ.

Classical Eilenberg theorem

Pseudovarieties of monoids

(Pseudo)varieties of finitely recognizable languages

Classes of finite monoids closed under homomorphic images (H), submonoids (S), and finite products ($\mathsf{P_f}$).

Operators $\mathcal{L}$ s.t. $\forall\ \Sigma$ and Γ finite:

$\mathcal{L}\Sigma$ is a set of f.r.l. on Σ.

$\mathcal{L}\Sigma$ is a BA w.r.t. $\cup$ and $\cap$.

Classical Eilenberg theorem

Pseudovarieties of monoids

Classes of finite monoids closed under homomorphic images (H), submonoids (S), and finite products (P_f).

(Pseudo)varieties of finitely recognizable languages

Operators $\mathcal{L}$ s.t. $\forall\ \Sigma$ and Γ finite:

$\mathcal{L}\Sigma$ is a set of f.r.l. on Σ.

$\mathcal{L}\Sigma$ is a BA w.r.t. $\cup$ and $\cap$.

$\mathcal{L}\Sigma$ is closed under derivatives.

Classical Eilenberg theorem

Pseudovarieties of monoids	**(Pseudo)varieties of finitely recognizable languages**
Classes of finite monoids closed under homomorphic images (H), submonoids (S), and finite products (P_f).	Operators $\mathcal{L}$ s.t. $\forall\ \Sigma$ and Γ finite: $\mathcal{L}\Sigma$ is a set of f.r.l. on Σ. $\mathcal{L}\Sigma$ is a BA w.r.t. $\cup$ and $\cap$. $\mathcal{L}\Sigma$ is closed under derivatives. Closure under morphic preimages: $\forall\ h : \Gamma^* \to \Sigma^*$, $L \in \mathcal{L}\Sigma \Rightarrow h^{-1}(L) \in \mathcal{L}\Gamma$

Classical Eilenberg theorem

Pseudovarieties of monoids $\Leftrightarrow$ **(Pseudo)varieties of finitely recognizable languages**

1-1 correspondence

Classes of finite monoids closed under homomorphic images (H), submonoids (S), and finite products (P_f).

Operators $\mathcal{L}$ s.t. $\forall\ \Sigma$ and Γ finite:

$\mathcal{L}\Sigma$ is a set of f.r.l. on Σ.

$\mathcal{L}\Sigma$ is a BA w.r.t. $\cup$ and $\cap$.

$\mathcal{L}\Sigma$ is closed under derivatives.

Closure under morphic preimages:
$\forall\ h : \Gamma^* \to \Sigma^*$, $L \in \mathcal{L}\Sigma \Rightarrow h^{-1}(L) \in \mathcal{L}\Gamma$

Generalization for algebras on a monad T

Consider the monad $\mathsf{T} = (T, \eta, \mu)$ on Set given by $T\Sigma = \Sigma^*$.
Then, $\mathrm{Alg}(\mathsf{T}) = \mathrm{Mon}$, the category of monoids.

Generalization for algebras on a monad T

Consider the monad $\mathsf{T} = (T, \eta, \mu)$ on Set given by $T\Sigma = \Sigma^*$.
Then, $\mathrm{Alg}(\mathsf{T}) = \mathrm{Mon}$, the category of monoids.

Pseudovariety of T–algebras: class of finite T–alg. closed under H, S, and $\mathsf{P_f}$.

Generalization for algebras on a monad T

Consider the monad T $= (T, \eta, \mu)$ on Set given by $T\Sigma = \Sigma^*$.
Then, $\mathrm{Alg}(\mathrm{T}) = \mathrm{Mon}$, the category of monoids.

Pseudovariety of T–algebras: class of finite T–alg. closed under H, S, and $\mathsf{P_f}$.

Alphabet

$\Sigma \in \mathrm{Set}$

Generalization for algebras on a monad $\mathbb{T}$

Consider the monad $\mathbb{T} = (T, \eta, \mu)$ on Set given by $T\Sigma = \Sigma^*$.
Then, $\mathrm{Alg}(\mathbb{T}) = \mathrm{Mon}$, the category of monoids.

Pseudovariety of $\mathbb{T}$–algebras: class of finite $\mathbb{T}$–alg. closed under H, S, and $\mathsf{P_f}$.

Alphabet

$\Sigma \in \mathrm{Set} \circlearrowleft \mathbb{T}$

Generalization for algebras on a monad T

Consider the monad T $= (T, \eta, \mu)$ on Set given by $T\Sigma = \Sigma^*$.
Then, $\mathrm{Alg}(\mathsf{T}) = \mathrm{Mon}$, the category of monoids.

Pseudovariety of T–algebras: class of finite T–alg. closed under H, S, and $\mathsf{P_f}$.

Language	Alphabet
$L \subseteq \Sigma^*$	$\Sigma \in \mathrm{Set} \circlearrowleft \mathsf{T}$

Generalization for algebras on a monad T

Consider the monad $\mathsf{T} = (T, \eta, \mu)$ on Set given by $T\Sigma = \Sigma^*$.
Then, $\mathrm{Alg}(\mathsf{T}) = \mathrm{Mon}$, the category of monoids.

Pseudovariety of T–algebras: class of finite T–alg. closed under H, S, and P_f.

Language

$$L \subseteq \Sigma^*$$

Alphabet

$$\Sigma \in \mathrm{Set} \circlearrowleft \mathsf{T}$$

Recognizable language

$\exists\ A \in \mathrm{Mon}$, $h \in \mathrm{Mon}(\Sigma^*, A)$ and $S \subseteq A$ s.t.

$$h^{-1}(S) = L$$

Generalization for algebras on a monad T

Consider the monad T $= (T, \eta, \mu)$ on Set given by $T\Sigma = \Sigma^*$.
Then, $\mathrm{Alg}(\mathrm{T}) = \mathrm{Mon}$, the category of monoids.

Pseudovariety of T**–algebras**: class of finite T–alg. closed under H, S, and $\mathsf{P_f}$.

Language

$L \subseteq T\Sigma$

Alphabet

$\Sigma \in \mathrm{Set}$ ⟲ T

Recognizable language

$\exists\ A \in \mathrm{Mon}$, $h \in \mathrm{Mon}(\Sigma^*, A)$ and $S \subseteq A$ s.t.

$$h^{-1}(S) = L$$

Generalization for algebras on a monad T

Consider the monad T $= (T, \eta, \mu)$ on Set given by $T\Sigma = \Sigma^*$.
Then, $\mathrm{Alg}(\mathsf{T}) = \mathrm{Mon}$, the category of monoids.

Pseudovariety of T–algebras: class of finite T–alg. closed under H, S, and P_f.

Language

$L \in \mathrm{Set}(T\Sigma, 2)$

Alphabet

$\Sigma \in \mathrm{Set} \circlearrowright \mathsf{T}$

Recognizable language

$\exists\ A \in \mathrm{Mon}$, $h \in \mathrm{Mon}(\Sigma^*, A)$ and $S \subseteq A$ s.t.

$$h^{-1}(S) = L$$

Generalization for algebras on a monad T

Consider the monad T $= (T, \eta, \mu)$ on Set given by $T\Sigma = \Sigma^*$.
Then, $\mathrm{Alg}(\mathrm{T}) = \mathrm{Mon}$, the category of monoids.

Pseudovariety of T**–algebras**: class of finite T–alg. closed under H, S, and $\mathrm{P_f}$.

Language

$L \in \mathrm{Set}(T\Sigma, 2)$

Alphabet

$\Sigma \in \mathrm{Set} \circlearrowleft \mathrm{T}$

Recognizable language

$\exists\ A \in \mathrm{Mon}$, $h \in \mathrm{Mon}(\Sigma^*, A)$ and $S \in \mathrm{Set}(A, 2)$ s.t.

$$h^{-1}(S) = L$$

Generalization for algebras on a monad T

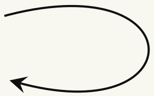
Consider the monad T $= (T, \eta, \mu)$ on Set given by $T\Sigma = \Sigma^*$.
Then, $\mathrm{Alg}(\mathrm{T}) = \mathrm{Mon}$, the category of monoids.

Pseudovariety of T**–algebras**: class of finite T–alg. closed under H, S, and $\mathrm{P_f}$.

Language

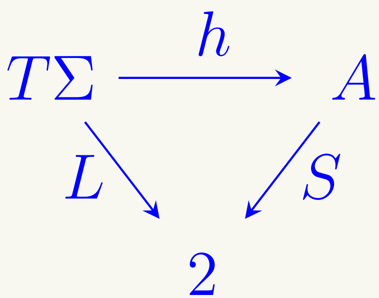
$L \in \mathrm{Set}(T\Sigma, 2)$

Alphabet

$\Sigma \in \mathrm{Set}$  T

Recognizable language

$\exists\ A \in \mathrm{Mon}$, $h \in \mathrm{Mon}(\Sigma^*, A)$ and $S \in \mathrm{Set}(A, 2)$ s.t.



Generalization for algebras on a monad T

How to define pseudovarieties of finitely recognizable T–languages?

Generalization for algebras on a monad T

How to define pseudovarieties of finitely recognizable T–languages?

Pseudovarieties of finitely recognizable languages

Operators $\mathcal{L}$ s.t. $\forall\ \Sigma$ and Γ finite:

$\mathcal{L}\Sigma$ is a set of f.r.l. on Σ.

$\mathcal{L}\Sigma$ is a BA w.r.t. $\cup$ and $\cap$.

$\mathcal{L}\Sigma$ is closed under derivatives.

Closure under morphic preimages:
$\forall\ h : \Gamma^* \to \Sigma^*, L \in \mathcal{L}\Sigma \Rightarrow h^{-1}(L) \in \mathcal{L}\Gamma$

Generalization for algebras on a monad T

How to define pseudovarieties of finitely recognizable T–languages?

Pseudovarieties of finitely recognizable languages

Operators $\mathcal{L}$ s.t. $\forall\ \Sigma$ and Γ finite:

$\mathcal{L}\Sigma$ is a set of f.r.l. on Σ. $\surd$

$\mathcal{L}\Sigma$ is a BA w.r.t. $\cup$ and $\cap$.

$\mathcal{L}\Sigma$ is closed under derivatives.

Closure under morphic preimages:
$\forall\ h : \Gamma^* \to \Sigma^*$, $L \in \mathcal{L}\Sigma \Rightarrow h^{-1}(L) \in \mathcal{L}\Gamma$

Generalization for algebras on a monad T

How to define pseudovarieties of finitely recognizable T–languages?

Pseudovarieties of finitely recognizable languages

Operators $\mathcal{L}$ s.t. $\forall\ \Sigma$ and Γ finite:

$\mathcal{L}\Sigma$ is a set of f.r.l. on Σ. $\surd$

$\mathcal{L}\Sigma$ is a BA w.r.t. $\cup$ and $\cap$.

$\mathcal{L}\Sigma$ is closed under derivatives.

Closure under morphic preimages:
$\forall\ h : T\Gamma \to T\Sigma$, $L \in \mathcal{L}\Sigma \Rightarrow L \circ h \in \mathcal{L}\Gamma$ $\surd$

How to define pseudovarieties of finitely recognizable T–languages?

Pseudovarieties of finitely recognizable languages

Operators $\mathcal{L}$ s.t. $\forall\ \Sigma$ and Γ finite:

$\mathcal{L}\Sigma$ is a set of f.r.l. on Σ. $\surd$

$\mathcal{L}\Sigma$ is a BA w.r.t. $\cup$ and $\cap$.

$\mathcal{L}\Sigma$ is closed under derivatives.

Closure under morphic preimages:
$\forall\ h : T\Gamma \to T\Sigma$, $L \in \mathcal{L}\Sigma \Rightarrow L \circ h \in \mathcal{L}\Gamma$ $\surd$

How to define pseudovarieties of finitely recognizable T–languages? Duality

Pseudovarieties of finitely recognizable languages

Operators $\mathcal{L}$ s.t. $\forall\ \Sigma$ and Γ finite:

$\mathcal{L}\Sigma$ is a set of f.r.l. on Σ. $\surd$

$\mathcal{L}\Sigma$ is a BA w.r.t. $\cup$ and $\cap$.

$\mathcal{L}\Sigma$ is closed under derivatives.

Closure under morphic preimages:
$\forall\ h : T\Gamma \to T\Sigma$, $L \in \mathcal{L}\Sigma \Rightarrow L \circ h \in \mathcal{L}\Gamma$ $\surd$

Generalization for algebras on a monad T

$$\mathrm{Set} \circlearrowleft \mathsf{T} = (T, \eta, \mu)$$

Generalization for algebras on a monad T

$$\mathrm{Set} \circlearrowleft \mathsf{T} = (T, \eta, \mu)$$

$$\Sigma$$

Generalization for algebras on a monad T

$$\mathrm{Set} \circlearrowleft \mathsf{T} = (T, \eta, \mu)$$

$$T\Sigma \qquad \Sigma$$

Generalization for algebras on a monad T

$$\mathrm{Set} \circlearrowleft \mathsf{T} = (T, \eta, \mu)$$

$$T\Sigma \xleftarrow{\;T\;} \Sigma$$

Generalization for algebras on a monad T

$$\mathrm{Set} \circlearrowleft \mathsf{T} = (T, \eta, \mu)$$

$$L \in \mathrm{Set}(T\Sigma, 2) \qquad T\Sigma \xleftarrow{\;T\;} \Sigma$$

Generalization for algebras on a monad T

$$\mathrm{Set} \circlearrowleft \mathsf{T} = (T, \eta, \mu)$$

$$L \in \mathrm{Set}(T\Sigma, 2) \xleftarrow{\ \mathrm{Set}(\_, 2)\ } T\Sigma \xleftarrow{\ T\ } \Sigma$$

Generalization for algebras on a monad T

$$\mathrm{Set} \circlearrowleft \mathsf{T} = (T, \eta, \mu)$$

$$G = \mathrm{Set}(\_, 2)$$

$$L \in \mathrm{Set}(T\Sigma, 2) \xleftarrow{\ \mathrm{Set}(\_,2)\ } T\Sigma \xleftarrow{\ T\ } \Sigma$$

Generalization for algebras on a monad T

$$\mathrm{CABA} \underset{G = \mathrm{Set}(\_,2)}{\overset{F = \mathrm{At}(\_)}{\rightleftarrows}} \mathrm{Set} \circlearrowleft \mathsf{T} = (T, \eta, \mu) \qquad (\cong)$$

$$L \in \mathrm{Set}(T\Sigma, 2) \xleftarrow{\mathrm{Set}(\_,2)} T\Sigma \xleftarrow{T} \Sigma$$

Generalization for algebras on a monad T

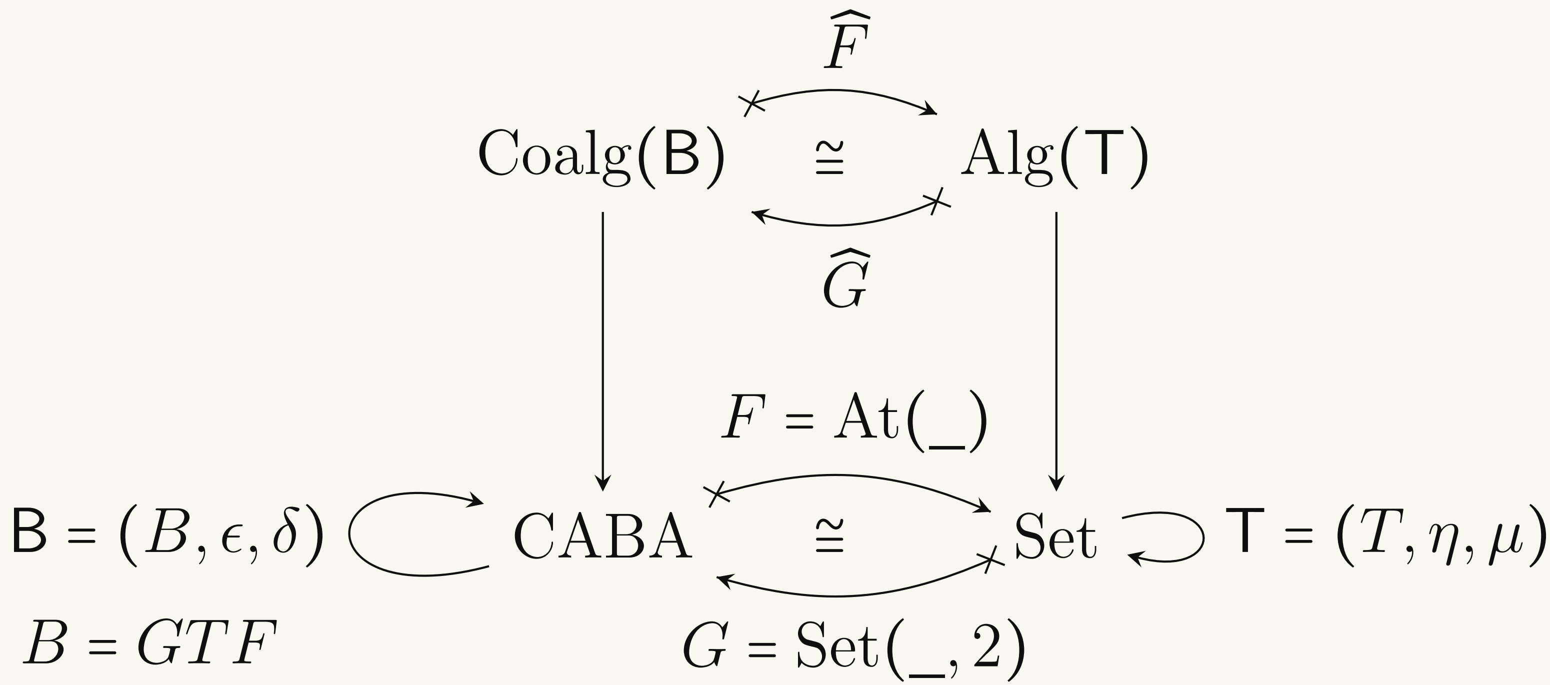
$$\mathsf{B} = (B, \epsilon, \delta) \circlearrowright \mathrm{CABA} \underset{G = \mathrm{Set}(\_, 2)}{\overset{F = \mathrm{At}(\_)}{\rightleftarrows}} \mathrm{Set} \circlearrowleft \mathsf{T} = (T, \eta, \mu)$$

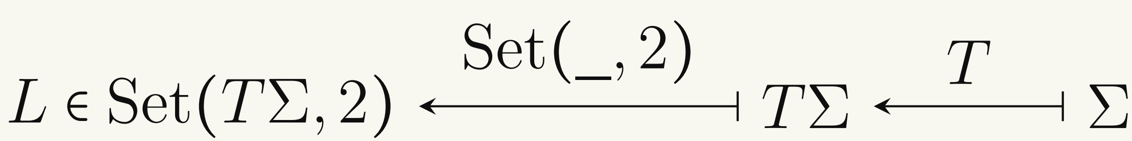
$$\mathrm{CABA} \cong \mathrm{Set}$$

$$B = GTF$$

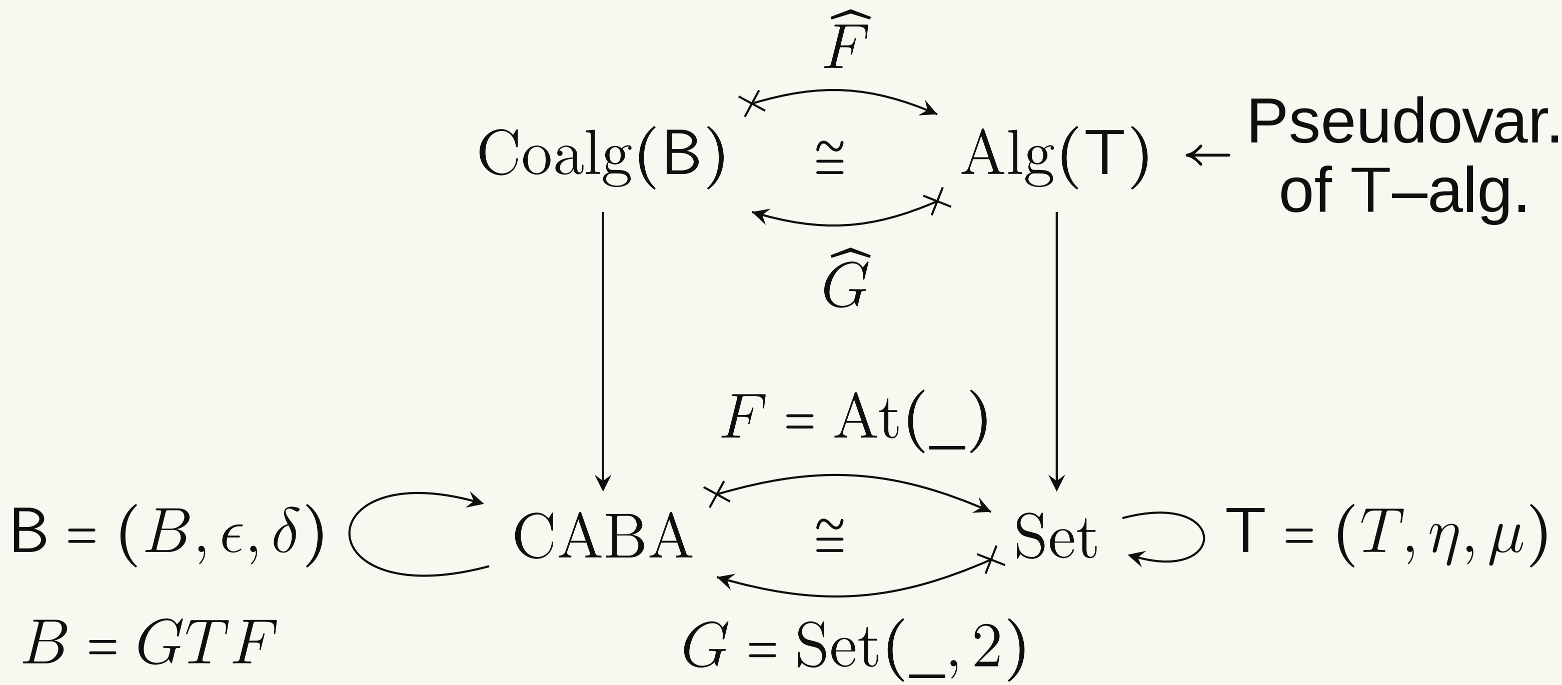
$$L \in \mathrm{Set}(T\Sigma, 2) \xleftarrow{\;\mathrm{Set}(\_, 2)\;} T\Sigma \xleftarrow{\;T\;} \Sigma$$

Generalization for algebras on a monad T



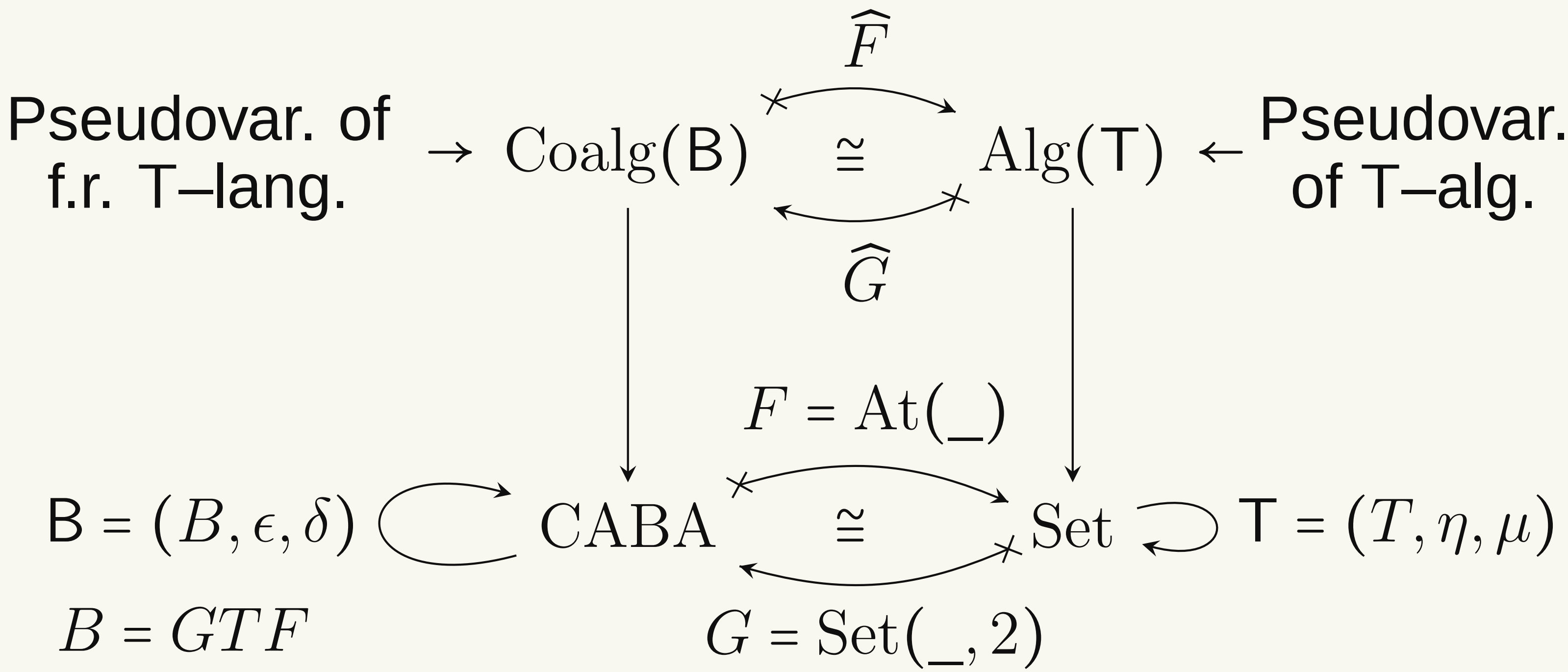


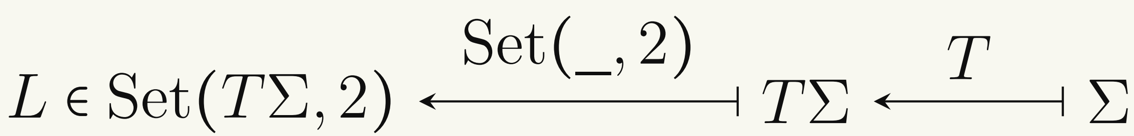
Generalization for algebras on a monad T



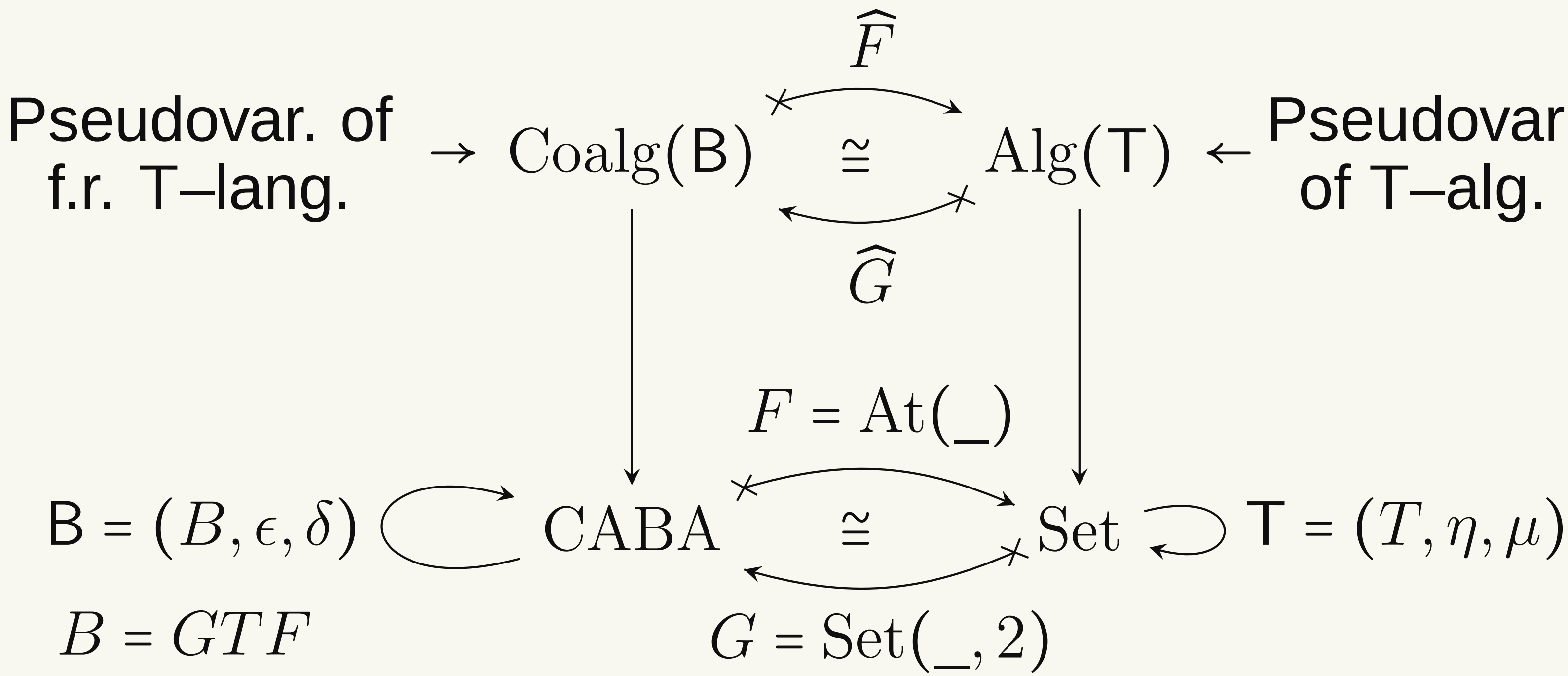
$$L \in \mathrm{Set}(T\Sigma, 2) \xleftarrow{\ \mathrm{Set}(\_,2)\ } T\Sigma \xleftarrow{\ T\ } \Sigma$$

Generalization for algebras on a monad T



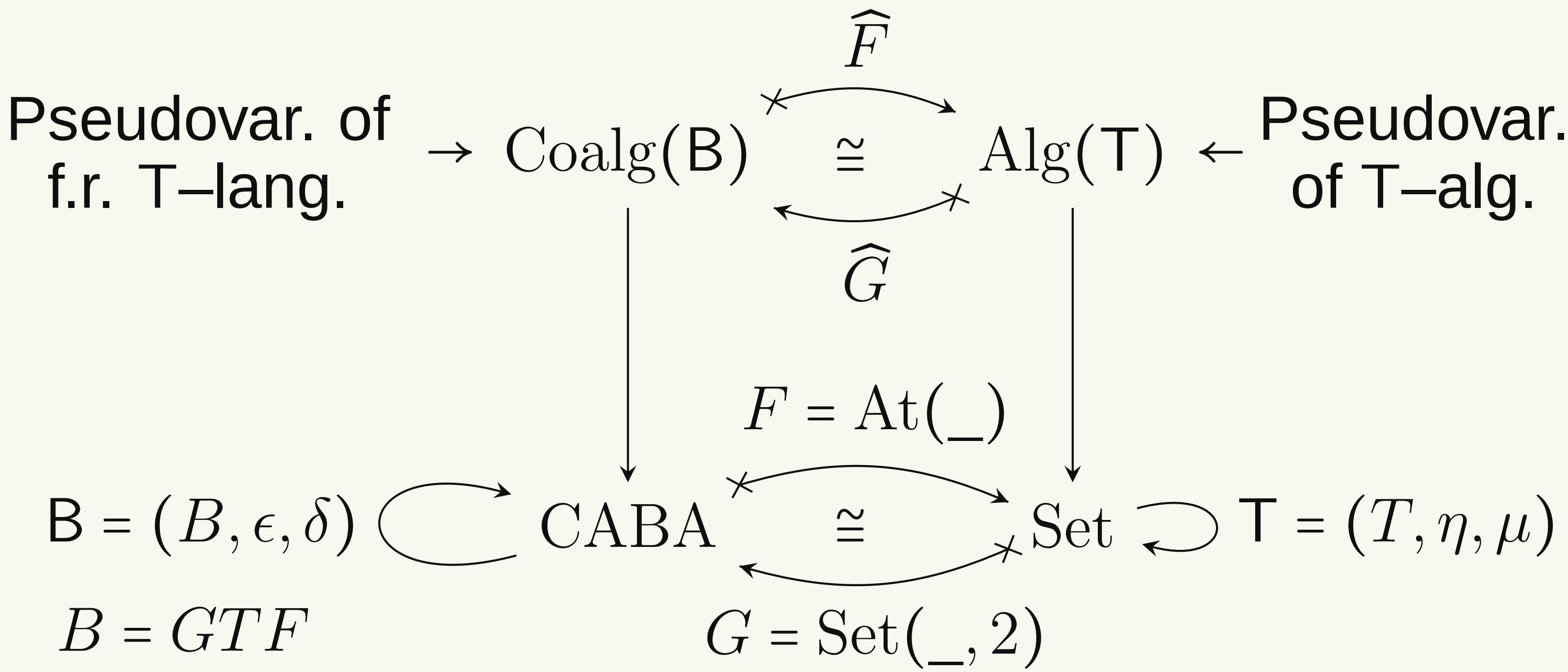


Generalization for algebras on a monad T



Pseudovariety of finitely recognizable T**-languages**

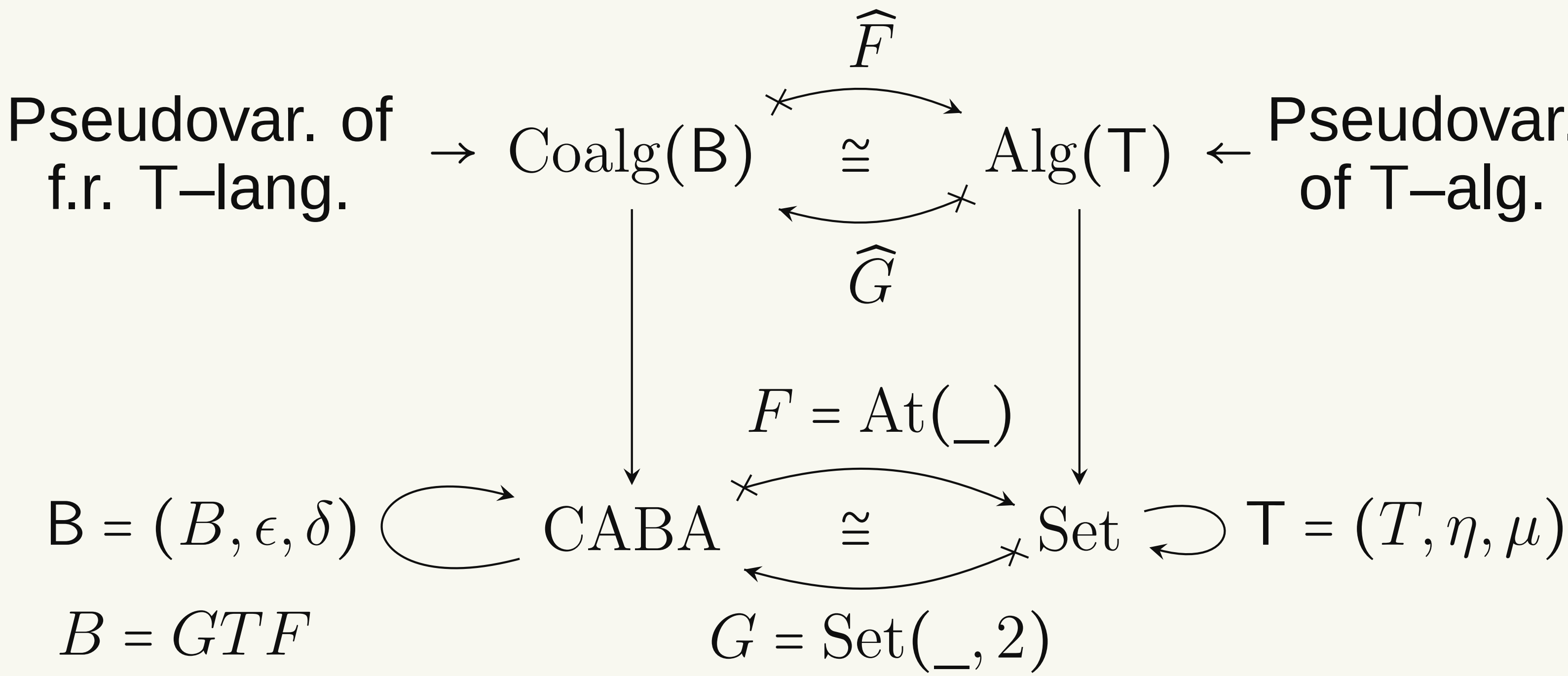
Generalization for algebras on a monad T



Pseudovariety of finitely recognizable T**-languages**

Operator $\mathcal{L}$ s.t. $\forall$ finite Σ, $\mathcal{L}\Sigma$ is a set of f.r. T-lang. on Σ and:

Generalization for algebras on a monad T



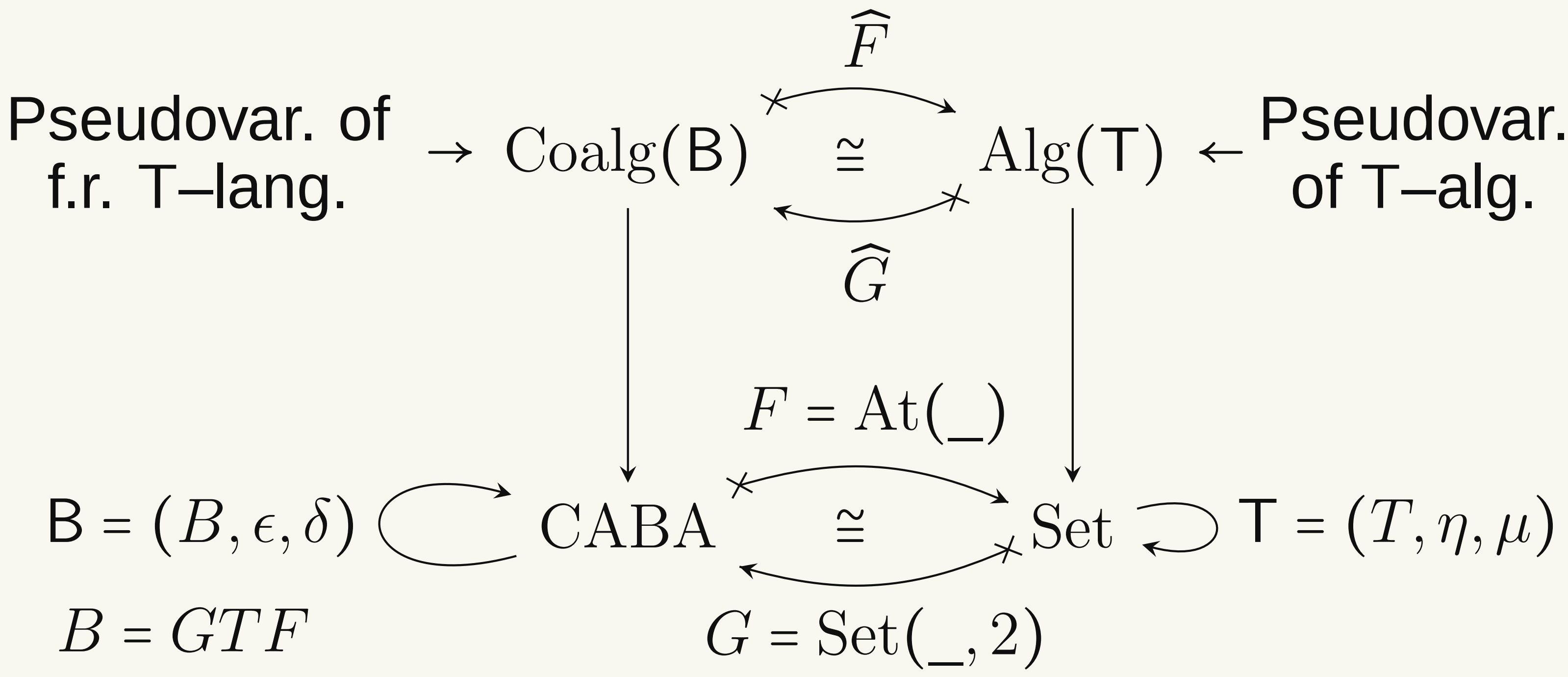
Pseudovariety of finitely recognizable T**-languages**

Operator $\mathcal{L}$ s.t. ∀ finite Σ, $\mathcal{L}\Sigma$ is a set of f.r. T-lang. on Σ and:

i) For every $L_1, L_2 \in \mathcal{L}\Sigma$, $\langle L_1, L_2 \rangle \subseteq \mathcal{L}\Sigma$.

($\langle L_1, L_2 \rangle$ subcoalg. of $\widehat{G}(T\Sigma, \mu_\Sigma)$ gen. by $\{L_1, L_2\}$.)

Generalization for algebras on a monad T



Pseudovariety of finitely recognizable T-languages

Operator $\mathcal{L}$ s.t. $\forall$ finite Σ, $\mathcal{L}\Sigma$ is a set of f.r. T-lang. on Σ and:

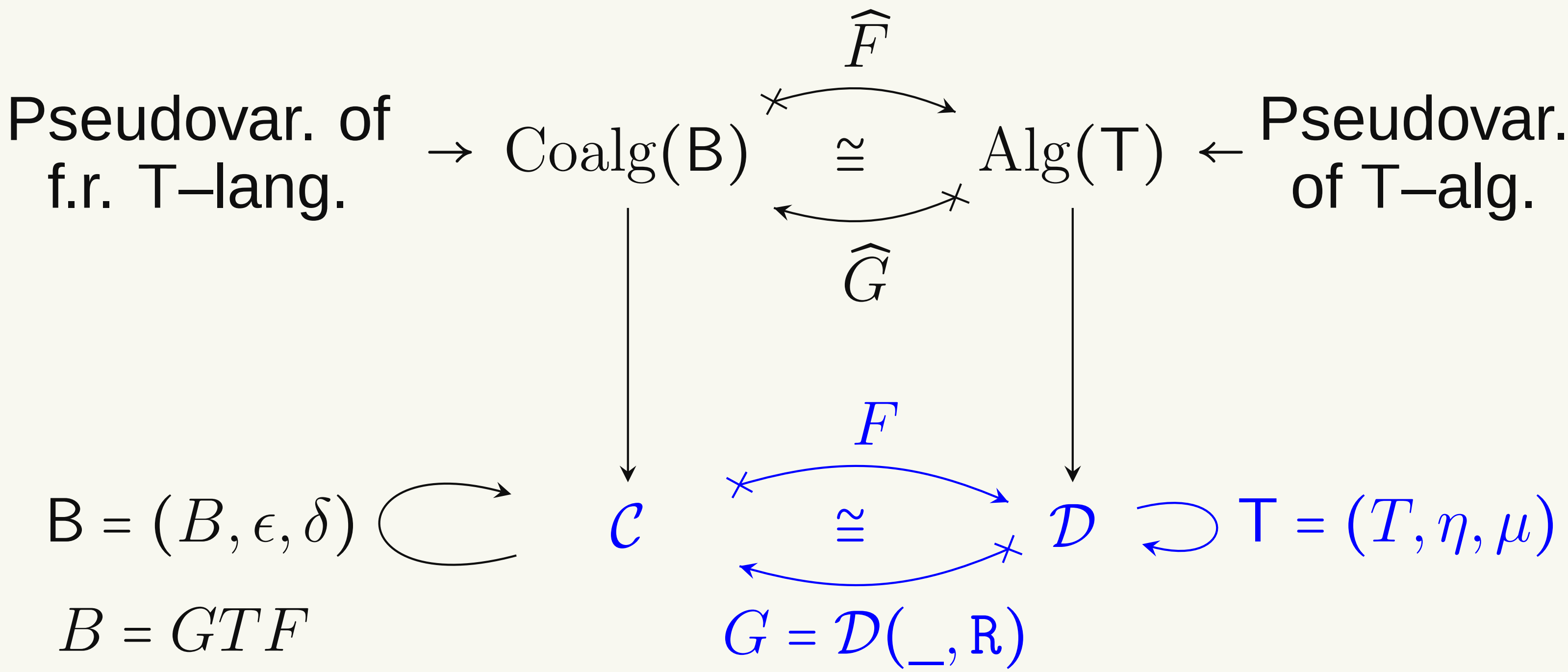
i) For every $L_1, L_2 \in \mathcal{L}\Sigma$, $\langle L_1, L_2 \rangle \subseteq \mathcal{L}\Sigma$.

($\langle L_1, L_2 \rangle$ subcoalg. of $\widehat{G}(T\Sigma, \mu_\Sigma)$ gen. by $\{L_1, L_2\}$.)

ii) $\mathcal{L}$ is closed under morphic preimages:

$\forall\ h \in \mathrm{Alg}(\mathsf{T})(T\Gamma, T\Sigma)$ and $L \in \mathcal{L}\Sigma$, $L \circ h \in \mathcal{L}\Gamma$.

Generalization for algebras on a monad T



Pseudovariety of finitely recognizable T-languages

Operator $\mathcal{L}$ s.t. ∀ finite Σ, $\mathcal{L}\Sigma$ is a set of f.r. T-lang. on Σ and:

i) For every $L_1, L_2 \in \mathcal{L}\Sigma$, $\langle L_1, L_2 \rangle \subseteq \mathcal{L}\Sigma$.

($\langle L_1, L_2 \rangle$ subcoalg. of $\widehat{G}(T\Sigma, \mu_\Sigma)$ gen. by $\{L_1, L_2\}$.)

ii) $\mathcal{L}$ is closed under morphic preimages:

$\forall\ h \in \mathrm{Alg}(\mathsf{T})(T\Gamma, T\Sigma)$ and $L \in \mathcal{L}\Sigma$, $L \circ h \in \mathcal{L}\Gamma$.

Generalization for algebras on a monad T

$$\mathcal{C} \underset{G = \mathcal{D}(\_,\, \mathtt{R})}{\overset{F}{\rightleftarrows}} \mathcal{D} \quad (\cong) \qquad \mathcal{D} \circlearrowleft \mathsf{T} = (T, \eta, \mu)$$

Generalization for algebras on a monad T

$$\mathcal{C} \underset{G = \mathcal{D}(\_,\, \mathtt{R})}{\overset{F}{\underset{\cong}{\rightleftarrows}}} \mathcal{D} \circlearrowleft \mathsf{T} = (T, \eta, \mu)$$

- $\mathcal{C}$ and $\mathcal{D}$ are concrete categories.

Generalization for algebras on a monad T

$$\mathcal{C} \underset{G = \mathcal{D}(\_,\mathtt{R})}{\overset{F}{\rightleftarrows}} \mathcal{D} \circlearrowleft \mathsf{T} = (T, \eta, \mu) \qquad (\cong)$$

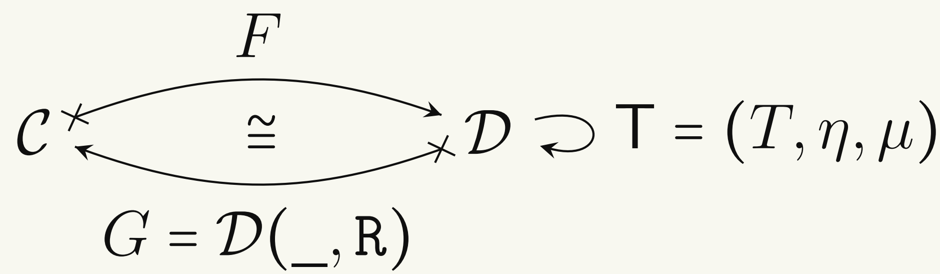
- $\mathcal{C}$ and $\mathcal{D}$ are concrete categories.
- $\mathcal{D}$ has gen. pushouts and T preserves them weakly.

Generalization for algebras on a monad T

$$\mathcal{C} \underset{G = \mathcal{D}(\_,\mathtt{R})}{\overset{F}{\rightleftarrows}} \mathcal{D} \circlearrowleft \mathsf{T} = (T, \eta, \mu) \qquad (\cong)$$

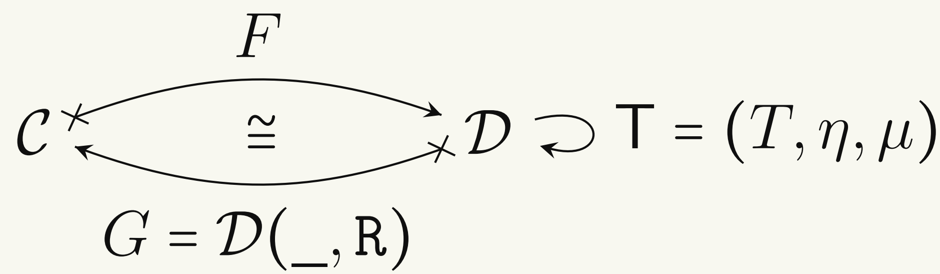
- $\mathcal{C}$ and $\mathcal{D}$ are concrete categories.
- $\mathcal{D}$ has gen. pushouts and T preserves them weakly.
- The underlying set of GX is $\mathcal{D}(X, \mathtt{R})$ and the def. of G on morphisms is as in $\mathcal{D}(\_, \mathtt{R})$.

Generalization for algebras on a monad T



- $\mathcal{C}$ and $\mathcal{D}$ are concrete categories.
- $\mathcal{D}$ has gen. pushouts and T preserves them weakly.
- The underlying set of GX is $\mathcal{D}(X, \mathtt{R})$ and the def. of G on morphisms is as in $\mathcal{D}(\_, \mathtt{R})$.
- $\mathcal{D}$ has epi–mono factorizations and epis are surjective.

Generalization for algebras on a monad T



- $\mathcal{C}$ and $\mathcal{D}$ are concrete categories.
- $\mathcal{D}$ has gen. pushouts and T preserves them weakly.
- The underlying set of GX is $\mathcal{D}(X, \mathtt{R})$ and the def. of G on morphisms is as in $\mathcal{D}(\_, \mathtt{R})$.
- $\mathcal{D}$ has epi–mono factorizations and epis are surjective.
- $\forall\ f \in \mathcal{D}(\Sigma, B)$, $e \in \mathrm{Alg}(\mathsf{T})(A, B)$ epi $\exists\ g \in \mathcal{D}(\Sigma, A)$ $e \circ g = f$.

Generalization for algebras on a monad T

$$\mathcal{C} \underset{G = \mathcal{D}(\_,\mathtt{R})}{\overset{F}{\rightleftarrows}} \cong \; \mathcal{D} \circlearrowleft \mathsf{T} = (T, \eta, \mu)$$

- $\mathcal{C}$ and $\mathcal{D}$ are concrete categories.
- $\mathcal{D}$ has gen. pushouts and T preserves them weakly.
- The underlying set of GX is $\mathcal{D}(X, \mathtt{R})$ and the def. of G on morphisms is as in $\mathcal{D}(\_, \mathtt{R})$.
- $\mathcal{D}$ has epi–mono factorizations and epis are surjective.
- $\forall\ f \in \mathcal{D}(\Sigma, B)$, $e \in \mathrm{Alg}(\mathsf{T})(A, B)$ epi $\exists\ g \in \mathcal{D}(\Sigma, A)$ $e \circ g = f$.
- $\mathcal{D}(X, \mathtt{R})$ separates points in X.

Generalization for algebras on a monad T

Under the previous assumptions we get an Eilenberg-like theorem for algebras on a monad T:

Pseudovarieties of T–algebras

Generalization for algebras on a monad T

Under the previous assumptions we get an Eilenberg-like theorem for algebras on a monad T:

Pseudovarieties of T–algebras

Classes of finite T–alg. closed under H, S, and P_f.

Generalization for algebras on a monad T

Under the previous assumptions we get an Eilenberg-like theorem for algebras on a monad T:

Pseudovarieties of T**–algebras**	**Pseudovarieties of f.r.** T**–languages**
Classes of finite T–alg. closed under H, S, and P_{f}.	

Generalization for algebras on a monad T

Under the previous assumptions we get an Eilenberg-like theorem for algebras on a monad T:

Pseudovarieties of T–algebras	**Pseudovarieties of f.r. T–languages**
Classes of finite T–alg. closed under H, S, and P_f.	Operators $\mathcal{L}$ s.t. $\forall\ \Sigma$ and Γ finite:

Generalization for algebras on a monad T

Under the previous assumptions we get an Eilenberg-like theorem for algebras on a monad T:

Pseudovarieties of T–algebras	**Pseudovarieties of f.r. T–languages**
Classes of finite T–alg. closed under H, S, and P_f.	Operators $\mathcal{L}$ s.t. $\forall$ Σ and Γ finite: $\mathcal{L}\Sigma$ is a set of f.r. T–lang. on Σ.

Generalization for algebras on a monad T

Under the previous assumptions we get an Eilenberg-like theorem for algebras on a monad T:

Pseudovarieties of T–algebras	**Pseudovarieties of f.r. T–languages**
Classes of finite T–alg. closed under H, S, and P_f.	Operators $\mathcal{L}$ s.t. $\forall$ Σ and Γ finite: $\mathcal{L}\Sigma$ is a set of f.r. T–lang. on Σ. $\forall$ $L_1, L_2 \in \mathcal{L}\Sigma$, $\langle L_1, L_2 \rangle \subseteq \mathcal{L}\Sigma$.

Generalization for algebras on a monad T

Under the previous assumptions we get an Eilenberg-like theorem for algebras on a monad T:

Pseudovarieties of T–algebras	**Pseudovarieties of f.r. T–languages**
Classes of finite T–alg. closed under H, S, and P_f.	Operators $\mathcal{L}$ s.t. $\forall\ \Sigma$ and Γ finite: $\mathcal{L}\Sigma$ is a set of f.r. T–lang. on Σ. $\forall\ L_1, L_2 \in \mathcal{L}\Sigma$, $\langle L_1, L_2 \rangle \subseteq \mathcal{L}\Sigma$. Closure under morphic preimages: $\forall\ h : T\Gamma \to T\Sigma$, $L \in \mathcal{L}\Sigma \Rightarrow L \circ h \in \mathcal{L}\Gamma$

Generalization for algebras on a monad T

Under the previous assumptions we get an Eilenberg-like theorem for algebras on a monad T:

Pseudovarieties of T–algebras	$\Leftrightarrow$ **1-1 correspondence**	**Pseudovarieties of f.r. T–languages**
Classes of finite T–alg. closed under H, S, and P_f.		Operators $\mathcal{L}$ s.t. $\forall\ \Sigma$ and Γ finite: $\mathcal{L}\Sigma$ is a set of f.r. T–lang. on Σ. $\forall\ L_1, L_2 \in \mathcal{L}\Sigma$, $\langle L_1, L_2 \rangle \subseteq \mathcal{L}\Sigma$. Closure under morphic preimages: $\forall\ h : T\Gamma \to T\Sigma$, $L \in \mathcal{L}\Sigma \Rightarrow L \circ h \in \mathcal{L}\Gamma$

Syntactic T-algebras

Given a T–language $L \in \mathcal{D}(T\Sigma, \mathtt{R})$ on $\Sigma \in \mathcal{D}$. The *syntactic T–algebra* of L is an algebra $(S_L, \alpha_L) \in \mathrm{Alg}(\mathsf{T})$ together with an epi $e_L \in \mathrm{Alg}(\mathsf{T})(T\Sigma, S_L)$ recognizing L s.t.:

Syntactic T-algebras

Given a T–language $L \in \mathcal{D}(T\Sigma, \mathtt{R})$ on $\Sigma \in \mathcal{D}$. The *syntactic* T–*algebra* of L is an algebra $(S_L, \alpha_L) \in \mathrm{Alg}(\mathsf{T})$ together with an epi $e_L \in \mathrm{Alg}(\mathsf{T})(T\Sigma, S_L)$ recognizing L s.t.:

- For every epi $e \in \mathrm{Alg}(\mathsf{T})(T\Sigma, A)$ recognizing L there exists $g \in \mathrm{Alg}(\mathsf{T})(A, S_L)$ s.t. $e_L = g \circ e$.

Syntactic T-algebras

Given a T–language $L \in \mathcal{D}(T\Sigma, \mathtt{R})$ on $\Sigma \in \mathcal{D}$. The *syntactic T–algebra* of L is an algebra $(S_L, \alpha_L) \in \mathrm{Alg}(\mathrm{T})$ together with an epi $e_L \in \mathrm{Alg}(\mathrm{T})(T\Sigma, S_L)$ recognizing L s.t.:

- For every epi $e \in \mathrm{Alg}(\mathrm{T})(T\Sigma, A)$ recognizing L there exists $g \in \mathrm{Alg}(\mathrm{T})(A, S_L)$ s.t. $e_L = g \circ e$.

Construction: S_L is the gen. pushout of the family $\{e_i : T\Sigma \to (A_i, \alpha_i)\}_{i \in I}$ of all epis in $\mathrm{Alg}(\mathrm{T})$ recognizing L.

Syntactic T-algebras

Given a T–language $L \in \mathcal{D}(T\Sigma, \mathtt{R})$ on $\Sigma \in \mathcal{D}$. The *syntactic* T–*algebra* of L is an algebra $(S_L, \alpha_L) \in \mathrm{Alg}(\mathsf{T})$ together with an epi $e_L \in \mathrm{Alg}(\mathsf{T})(T\Sigma, S_L)$ recognizing L s.t.:

- For every epi $e \in \mathrm{Alg}(\mathsf{T})(T\Sigma, A)$ recognizing L there exists $g \in \mathrm{Alg}(\mathsf{T})(A, S_L)$ s.t. $e_L = g \circ e$.

Construction: S_L is the gen. pushout of the family $\{e_i : T\Sigma \to (A_i, \alpha_i)\}_{i \in I}$ of all epis in $\mathrm{Alg}(\mathsf{T})$ recognizing L.

Equivalently, S_L is the dual algebra of the B–coalgebra $\langle L \rangle$ and e_L is the dual of the inclusion B-coalgebra morphism $i_L : \langle L \rangle \to G(T\Sigma)$.

Conclusions

- We proved a categorical generalization of Eilenberg-type theorem for algebras on a monad.

Conclusions

- We proved a categorical generalization of Eilenberg-type theorem for algebras on a monad.
- A categorical definition of pseudovarieties of finitely recognizable T–languages was given.

Conclusions

- We proved a categorical generalization of Eilenberg-type theorem for algebras on a monad.
- A categorical definition of pseudovarieties of finitely recognizable T–languages was given.
- An categorical construction of syntactic T–algebras by using generalized pushouts was given.

Conclusions

- We proved a categorical generalization of Eilenberg-type theorem for algebras on a monad.
- A categorical definition of pseudovarieties of finitely recognizable T–languages was given.
- An categorical construction of syntactic T–algebras by using generalized pushouts was given.

Future work:

Conclusions

- We proved a categorical generalization of Eilenberg-type theorem for algebras on a monad.
- A categorical definition of pseudovarieties of finitely recognizable T–languages was given.
- An categorical construction of syntactic T–algebras by using generalized pushouts was given.

Future work:

- Consider predual categories instead of dual categories.

Conclusions

- We proved a categorical generalization of Eilenberg-type theorem for algebras on a monad.
- A categorical definition of pseudovarieties of finitely recognizable T–languages was given.
- An categorical construction of syntactic T–algebras by using generalized pushouts was given.

Future work:

- Consider predual categories instead of dual categories.
- Drop finiteness property in the definition of pseudovarieties.

Conclusions

- We proved a categorical generalization of Eilenberg-type theorem for algebras on a monad.
- A categorical definition of pseudovarieties of finitely recognizable T–languages was given.
- An categorical construction of syntactic T–algebras by using generalized pushouts was given.

Future work:

- Consider predual categories instead of dual categories.
- Drop finiteness property in the definition of pseudovarieties.
- Connection with Reiterman's theorem.

Conclusions

- We proved a categorical generalization of Eilenberg-type theorem for algebras on a monad.
- A categorical definition of pseudovarieties of finitely recognizable T–languages was given.
- An categorical construction of syntactic T–algebras by using generalized pushouts was given.

Future work:

- Consider predual categories instead of dual categories.
- Drop finiteness property in the definition of pseudovarieties.
- Connection with Reiterman's theorem.
- Find more examples.