On local characterization of global timed bisimulation of abstract continuous-time systems

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Bisimulation relation on states of LTS

- Q is a set of states of a labeled transition system (LTS)
- $\stackrel{a}{\to}$ is a transition relation (a is a label).

$R \subseteq Q \times Q$ is a bisimulation, if

- $(q_1, q_2) \in R$ implies that for each $q_1' \in Q$ and a such that $q_1 \stackrel{a}{\to} q_1'$ there exists $q_2' \in Q$ such that $q_2 \stackrel{a}{\to} q_2'$ and $(q_1', q_2') \in R$
- $(q_1, q_2) \in R$ implies that for each $q_2' \in Q$ and a such that $q_2 \stackrel{a}{\to} q_2'$ there exists $q_1' \in Q$ such that $q_1 \stackrel{a}{\to} q_1'$ and $(q_1', q_2') \in R$.

Local character: expressed in terms of immediate transitions

Main question of this talk

- LTS can be considered as a discrete time a dynamical system.
- **Can we adapt the notion of a bisimulation relation on states of an LTS to continuous-time dynamical systems and preserve its local character ?**
- Examples of continuous-time systems: systems described by differential equations, differential inclusions, switched systems, certain classes of hybrid (discrete-continuous) systems, etc.
- But we don't want to adapt to each class of systems separately. We want one sufficiently general definition that works for many of them.

Continuous- or hybrid-time models

Examples:

- **Differential equation** $x'(t) = f(t, x(t))$.
  Trajectory: $t \mapsto x(t)$
- **Switched system** $x'(t) = f_{q(t)}(x(t))$, $q(t) \in \{1, 2, ..., n\}$ is a piecewise-constant signal. Trajectory: concatenation of solutions of $x'(t) = f_{q(t)}(t)$ on intervals where q is constant
- **Hybrid (discrete-continuous) dynamical system**, etc.

Each has an associated set of **states** and **trajectories** / executions.

Trajectories

Most continuous-time models can be described using the notion of a trajectory (but there may be no notion of a "transition"):

- $(T, \leq)$ is a totally ordered **time domain**
- $Q \neq \emptyset$ is a **state-space**.
- Tr is a set of **(partial) trajectories** $s : T \tilde{\rightarrow} Q$ such that $dom(s)$ is an interval in T.

Example

- Let the model M be

$$y'(t) = -y(t),$$

y is real-valued, t is real time

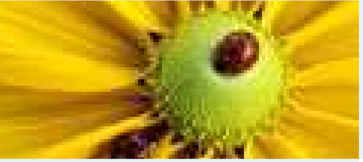
- For any initial time $t = t_0 \in \mathbb{R}$ and any state $y_0 \in \mathbb{R}$ we can define a trajectory that starts at t_0 and is defined for all $t \geq t_0$:

$$y(t) = y_0 e^{-(t-t_0)}$$

Well-known approaches

Approach 1 (Reduction to LTS). Associate some LTS L with M and say that a bisimulation for L is a bisimulation for M.

- E.g. as M take an ODE with input $y'(t) = f(y(t), u(t))$, y and u are real-valued functions, $y(t)$ is a state, $u(t)$ is an input control.
- As L take an LTS where states and labels are real numbers and $q_1 \xrightarrow{a} q_2$ iff there is u and a solution y s.t. $y(0) = q_1$ and $y(a) = q_2$
- See e.g. *G.J. Pappas "Bisimilar linear systems", Automatica, 2003*



Issues

- **Not exactly what we want**: bisimulation is defined in terms of immediate transitions in the LTS, but transitions express some **long-term reachability** on the states of M.
- Rarely explicit expressions of trajectories (solutions) for all t are known / can be found.
- But there are many methods of **local** (short-term) analysis of behavior of M, e.g. linearization of ODEs, Hartman-Grobman theorem, series expansions and approximations of solutions, singularity analysis.
- No obvious way to **apply such methods**, if we want to prove that R **is a bisimulation**.

Well-known approaches

Approach 2 (Direct definition).

- No LTS. Define bisimulation for some class of continuous-time systems directly in terms of its trajectories (paths, executions, etc. what name is more appropriate for the given class).
- Example: *J. Davoren, P. Tabuada. On simulations and bisimulations of general flow systems. HSCC 2007.*
- The **issue** is the same: the definition of bisimulation is about long-term behaviors of a continuous-time model.

Other examples can be found in the paper.

Definition by J. Davoren, P. Tabuada

If Φ_1, Φ_2 *are* **general flow systems** *over value spaces* X_1, X_2 *with the same time line, a binary relation* R *between* X_1, X_2 *is a* **timed simulation** *of* Φ_1 *by* Φ_2*, if* $dom(\Phi_1) \subseteq dom(R)$ *and for all* $x_1, x_1' \in X_1$*,* $x_2 \in X_2$ *such that* $(x_1, x_2) \in R$ *and for all times* $t > 0$*, if there is* **a path** *(!)* $\gamma_1 \in \Phi_1(x_1)$ *such that* $x_1' = \gamma_1(t)$*, then there is* $x_2' \in X_2$ *and a path* $\gamma_2 \in \Phi_2(x_2)$ *such that* $x_2' = \gamma_2(t)$*,* $dom(\gamma_2) = dom(\gamma_1)$*, and* $(\gamma_1(s), \gamma_2(s)) \in R$ *for all* $s \in dom(\gamma_2) \cap [0, t]$*.*
A relation R *is a* **timed bisimulation** *between* Φ_1*,* Φ_2*, if both* R *and* R^{-1} *are timed simulations.*

We propose the following approach:

(1) Consider a class of **abstract continuous-time models** of about the same level of abstraction as LTS such that many well-known continuous-time models can be considered as concretizations of systems of this class.

(2) However, we do not aim at absolute generality. We want a class of models for which checking global-in-time properties can be **reduced** to checking local-in-time properties

General flow systems and various related notions satisfy (1), but seem to be too general for (2).

We propose the following approach:

(3) Adapt a direct definition of bisimulation e.g. by J. Davoren and P. Tabuada to the class of models (1). Call the defined notion a **global timed bisimulation**.

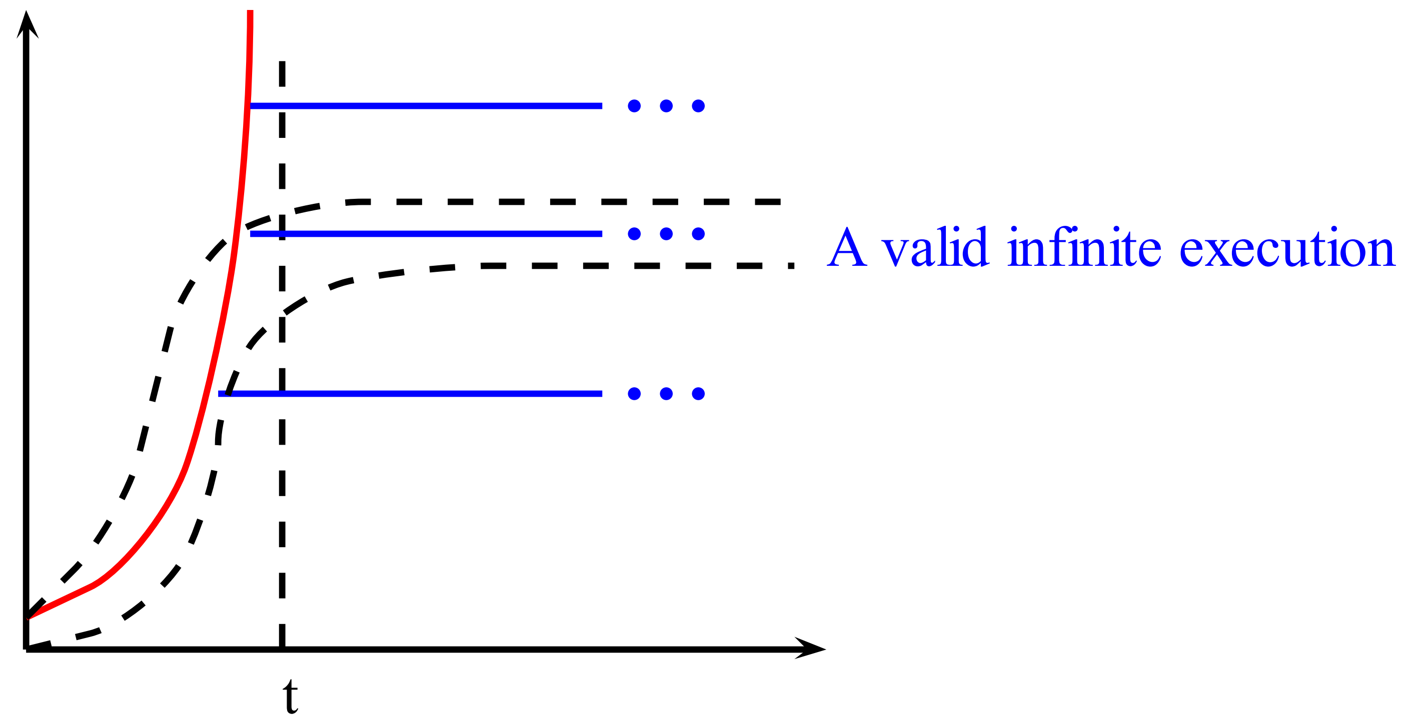
(4) Find and prove an expression of global timed bisimulation in terms of local-in-time behaviors of a system.
We will call this expression a **local characterization of global timed bisimulation**.

More on the class of models

Note that using the class (1) we want to cover both

- **non-deterministic** continuous-time models (more than one trajectory starting in the same state)
  e.g. trajectories are $(y(t), z(t))$,
  $y'(t) = z(t) \wedge z(t) \in [0, 1]$
- models that are **not forward-complete** (some partial trajectories may have no extension to the whole time domain, e.g. be Zeno-like or lead to finite-time blow-ups). E.g.
  $y'(t) = y^2(t)$.

Illustration



Potential difficulties

- An implementation of the approach must be able to express in "local terms", that if $(q_1, q_2) \in R$ and there is a trajectory starting at q_1 of the duration t, then there is a trajectory **of the duration** t starting at q_2.
- By "local terms" we mean a condition that can be checked agains a model M in each arbitrarily small neighborhood of each time moment $(t - \epsilon, t + \epsilon)$ independently.
- That is it should be able to express the existence of **long trajectories** in terms of the existence of some **arbitrarily short** trajectories (without forward-completeness).

Comparison to the discrete-time case

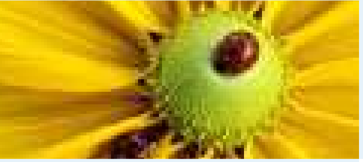
- **Why the mentioned difficulty is a non-issue in discrete-time case ?**
- For LTS, if for there is an outgoing transition from each state, then there is infinite run (of length ω) starting at each state.
- In continuous-time, if there is short trajectory starting at each state, but we impose no uniform lower bound on their time length, there is no guarantee that there will be infinite (in time) trajectories [e.g.: $y'(t) = y^2(t)$].
  If we impose uniform lower bound, our condition which implies the existence of infinite trajectories is not "local".

In discrete-time case

- Let $s_1 = q_1 \xrightarrow{a} q_2 \xrightarrow{b} q_3 \xrightarrow{c} q_4 \ldots \xrightarrow{y} q_n$ be a finite run of an LTS L.
- Then it can be extended e.g. to $s_2 = q_1 \xrightarrow{a} q_2 \xrightarrow{c} q_3 \ldots \xrightarrow{y} q_n \xrightarrow{z} q_{n+1}$ and so on.
- Then we can build an infinite chain of extensions (ordered by prefix relation) $s_1 \prec s_2 \prec s_3 \prec \ldots$.
- The limit (in the set of all possible finite and infinite runs ordered by $\prec$) is a run of L.

In continuous-time case

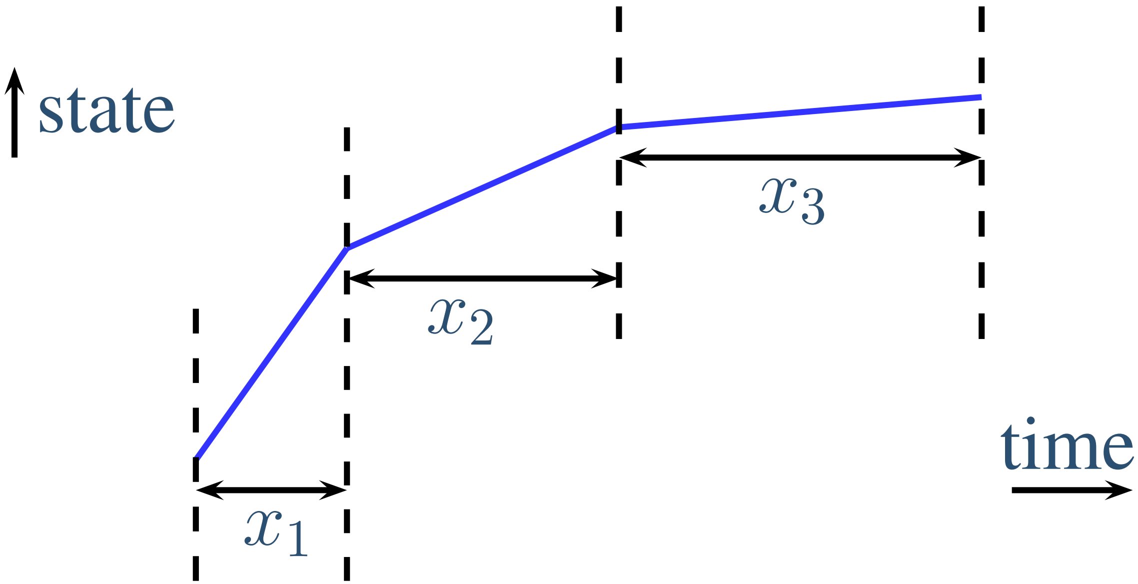
- Let $s_1 : [0, t_1] \to \mathbb{R}$ be a trajectory of M.
- Under assumption of existence of a trajectory starting at each state, we can extend s_1 to some $s_2 : [0, t_2] \to \mathbb{R}$ with $t_2 > t_1$ (just continue from the state $s_1(t_1)$), and so on. Then we can build an infinite chain of extensions (ordered by prefix relation) $s_1 \prec s_2 \prec \ldots$.
- Even if we assume that the set of trajectories of M is ω-complete or chain complete, we cannot conclude that M has infinite-in-time trajectories (maybe $\lim_{n \to \infty} t_n < \infty$).



Lower bounds on lengths of extensions

- To make sure that the sequence of extensions $s_1, s_2, \ldots$ grows fast enough, we can introduce a lower bound on the "size" of the difference $dom(s_{i+1}) \backslash dom(s_i)$.
- For example, if $dom(s_i) = [0, b_i] \subset \mathbb{R}_+$ and
  $b_{i+1} - b_i > f(b_i)$
  where f is a continuous function such that
  $f(x) > 0$ for all $x > 0$,
  then $b_i \to +\infty$.

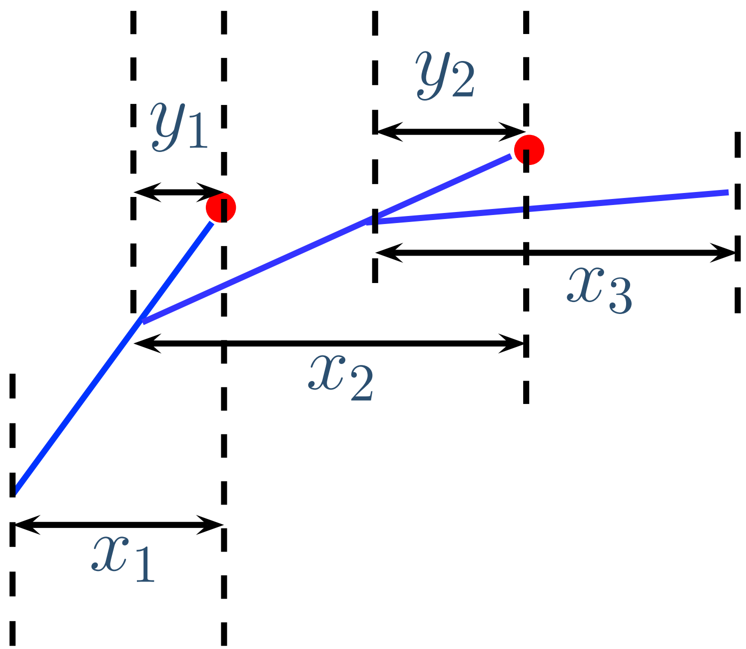
Monotone reasoning



$$x_{n+1} \geq f(x_1 + x_2 + \ldots + x_n)$$

$f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, $f(x) > 0$ if $x > 0$. Then $x_1 + x_2 + x_3 + \ldots = +\infty$.

We propose non-monotone reasoning



$$x_{n+1} \geq y_n + f(y_n)$$

$f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous monotone function, $f(x) > 0$ if $x > 0$.

The proposed class of models

We consider the following class of abstract models that satisfy our requirements (1) and (2):

- A **non-deterministic complete Markovian*** **system (NCMS)** as a continuous-time analog of a time-labeled transition system (transitions "merge" into a "flow")
- Concretizations include ODEs and switched (discrete-continuous) systems (details are in the paper).

* This name comes from the theory of stochastic processes. In our case, it is used in the context of pure non-determinism

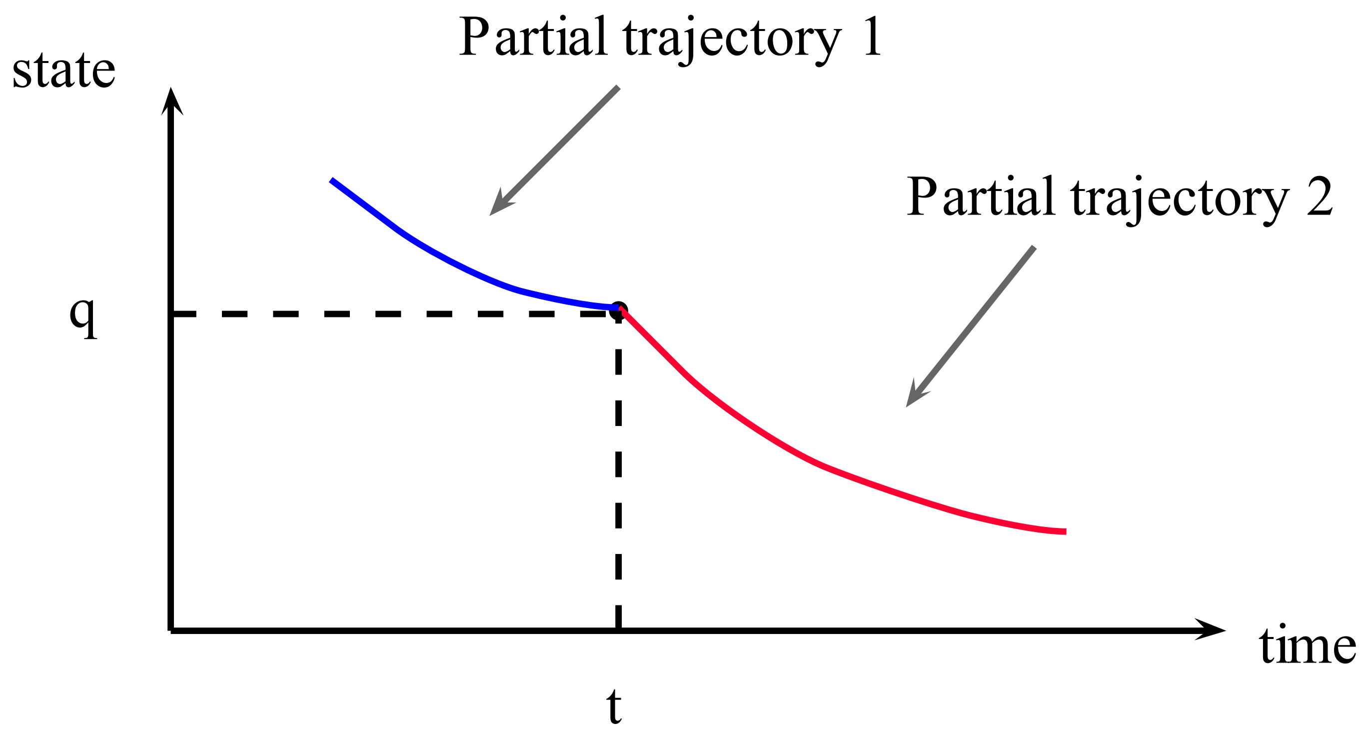
Definition

Let Q be a set of **states**,
$T = [0, +\infty)$ is the (real) **time scale**,
$\mathfrak{T}$ is the set of non-empty, non-singleton connected subsets (intervals) of T
A NCMS is defined by a set of **partial trajectories**

$$E \subseteq \{\tilde{s} : A \to Q \,|\, A \in \mathfrak{T}\}$$

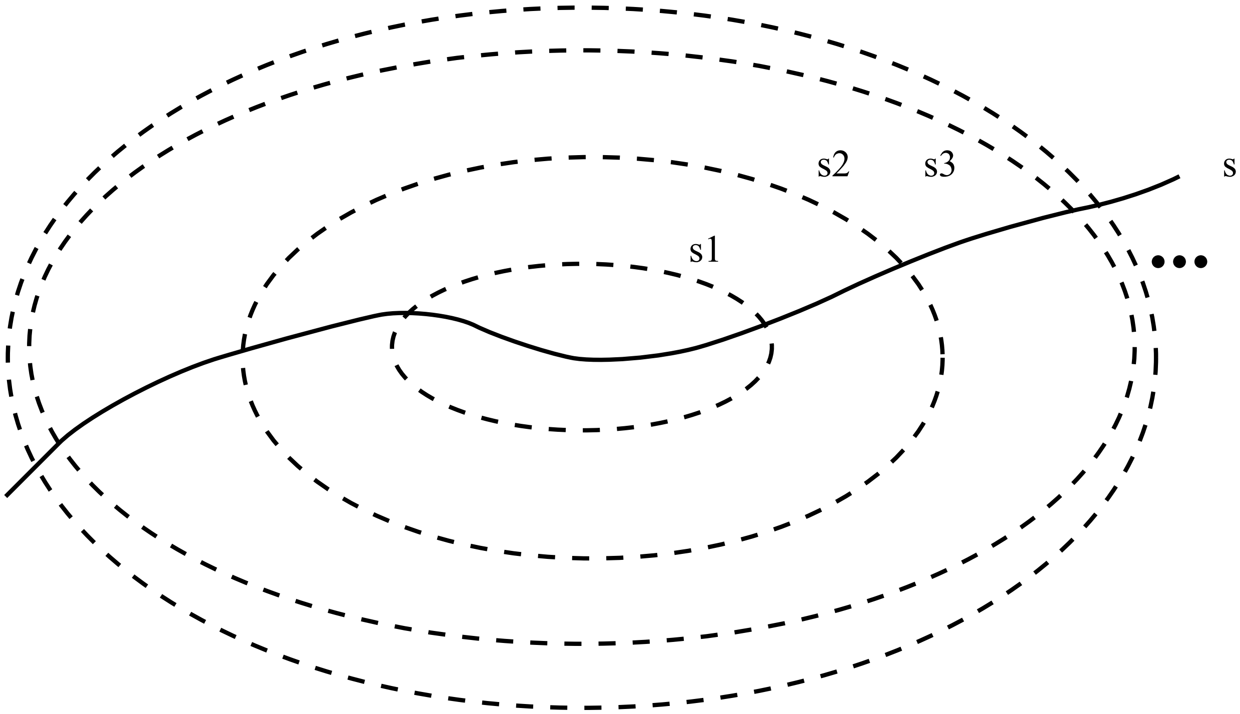
- E is closed under proper **restrictions**: if $\tilde{s} \in E$, $B \in \mathfrak{T}$, $B \subseteq dom(\tilde{s})$, then $\tilde{s}|_B \in E$, i.e. *a non-trivial part of a non-trivial partial trajectory is a non-trivial partial trajectory*

Markov-like property



Completeness

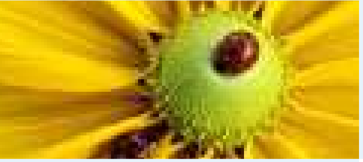
Each non-empty chain of trajectories in Tr has a supremum in Tr with respect to the **subtrajectory** relation (s_1 is a subtrajectory of s_2, if the graph of s_1 is a subset of the graph of s_2).



if s1,s2,s3... are partial trajectories then s is a partial trajectory

Continuous-time analog of LTS

- In the case of LTS, labels are some data associated with transitions and traces are sequences of labels along runs of an LTS.
- In NCMS the role of "transitions" play "infinitesimally short trajectories" and "labels" are certain values associated with such trajectories.



Formalization: labeled NCMS

A function λ on a CPR set of trajectories Tr is a **trace**, if the following conditions hold:

(1) (Preservation of domain) For each $s \in Tr$, $\lambda(s)$ is a function defined on $dom(s)$.

(2) (Monotonicity) If $s_1, s_2 \in Tr$ and $s_1 \sqsubseteq s_2$, then $\lambda(s_1) \sqsubseteq \lambda(s_2)$.

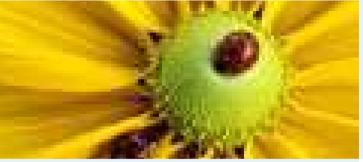
A labeled NCMS is a pair (Σ, λ), where $\Sigma = (T, Q, Tr)$ is a NCMS and λ is trace on Tr.

Extensibility measures

- A **right extensibility measure** is a function $f^+ : T \times T \tilde{\to} T$ which is continuous on $\{(x, y) \in T \times T \,|\, x \leq y\}$ and
  - $f^+(x, y)$ is strictly decreasing in x and strictly increasing in y;
  - $f^+(x, x) = x$.
- A **right dead-end path** is a trajectory $s : A \to Q$ such that A has a form $[a, b)$, where $a, b \in T$, and there is no $s' : [a, b') \to Q \in Tr$ such that $b' > b$ and $s = s'|_{dom(s)}$.

Escapes

- An **escape** from a right dead-end path $s : [a, b) \to Q$ is a trajectory $s' : [c, d] \to Q$ where $d < +\infty$, or $s' : [c, d) \to Q$, where $d = +\infty$, such that $c \in (a, b)$, $d > b$, $s(c) = s'(c)$.
- A right dead-end path $s : [a, b) \to Q$ is f^+-**escapable**, if there exists an escape $s' : [c, d] \to Q$ from s such that $d \geq f^+(c, b)$.

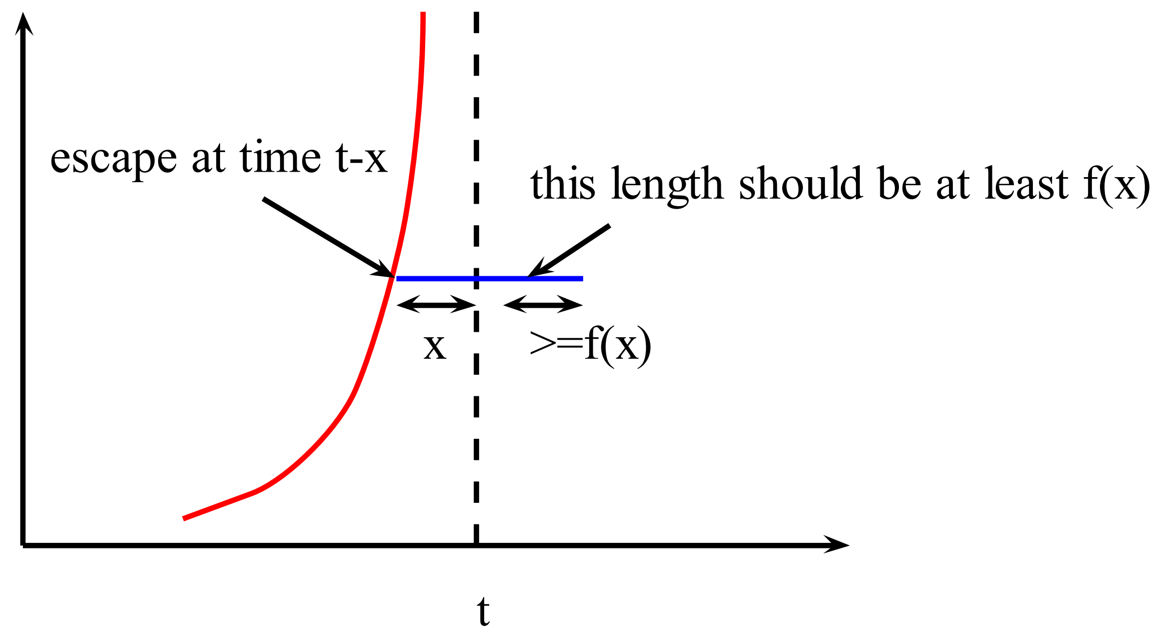


Our main tools

- NCMS has **local forward extensibility (LFE)** property, if each trajectory $s : [a, b] \to Q$ can be extended to $s' : [a, b'] \to Q$, $b' > b$.

Theorem. Let Σ be a NCMS which has LFE and f^+ be a normal right extensibility measure (e.g. $f^+(x, y) = 2y - x$). Each right dead-end path in Σ has an infinite escape iff each right dead-end path is f^+-escapable.

A special case of the theorem



Right extensibility measure
$f^+(x,y) = y + f(y-x)$.
where $f : T \to T$ is a monotone continuous function s.t. $f(x) > 0$, if $x > 0$ and $f(0) = 0$ (e.g. $f(x) = x$).

Auxiliary notions

Let $s_1, s_2 : T \tilde{\to} Q$ and $R \subseteq Q \times Q$ be a binary relation. Then the functions s_1 and s_2 are:

(1) pointwise in R, if $dom(s_1) = dom(s_2)$ and $(s_1(t), s_2(t)) \in R$ for $t \in dom(s_1)$;

(2) pointwise in R on a set $A \subseteq T$, if $A \subseteq dom(s_1) \cap dom(s_2)$ and $(s_1(t), s_2(t)) \in R$ for all $t \in A$;

(3) pointwise in R in a right neighborhood of $t \in T$, if there exists $t' > t$, such that s_1, s_2 are pointwise in R on $[t, t')$;

(4) pointwise in R in a deleted left neighborhood of $t \in T$, if $t > 0$ and there is $t' \in [0, t)$ such that s_1, s_2 are pointwise in R on (t', t).

Global timed bisimulation

- A relation $R \subseteq Q \times Q$ is a **global timed simulation** on a labeled NCMS (Σ, λ), if for each $(q_1, q_2) \in R$ and $s_1 \in Tr$ which starts at q_1 there is $s_2 \in Tr$ which starts at q_2, $\lambda(s_1) = \lambda(s_2)$, and s_1, s_2 are pointwise in R.
- A relation $R \subseteq Q \times Q$ is a **local timed simulation** on a labeled NCMS (Σ, λ), if for each $(q_1, q_2) \in R$, $s_1 \in Tr$, and $t_0 \in T$ which starts in q_1 at time t_0 there exists $s_2 \in Tr$ which starts in q_2 at time t_0, and $\lambda(s_1)$, $\lambda(s_2)$ are pointwise equal in a right neighborhood of t_0, and s_1, s_2 are pointwise in R in a right neighborhood of t_0.

Main defintion

A relation $R \subseteq Q \times Q$ is a f^+**-timed simulation** on a labeled NCMS (Σ, λ), if R is a local timed simulation on (Σ, λ) and for each $s_1, s_2 \in Tr$ and $t_0 \in dom(s_1)$ such that s_1, s_2 are pointwise in R in a deleted left neighborhood of t_0, $\lambda(s_1)$, $\lambda(s_2)$ are pointwise equal on $[t_0', t_0)$ for some $t_0' < t_0$, there exist $s_2' \in Tr$, $t_1 \in dom(s_2) \cap dom(s_2')$, and $t_2 \in T$ such that
(1) $t_1 < t_0$ and $s_2(t_1) = s_2'(t_1)$;
(2) either $t_2 \geq f^+(t_1, t_0)$, or t_2 is the maximal element of $dom(s_1)$;
(3) $\lambda(s_1) \doteq_{[t_1, t_2]} \lambda(s_2')$;
(4) s_1 and s_2' are pointwise in R on $[t_1, t_2]$.

Main result

Theorem. A relation $R \subseteq Q \times Q$ is a global timed bisimulation on (Σ, λ) if and only if R is a f^+-timed bisimulation on (Σ, λ).