Regular Behaviours with Names

On the Rational Fixpoint of Endofunctors on Nominal Sets

Stefan Milius, Lutz Schröder, **Thorsten Wißmann**

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Automata, Trees, . . .

involving

Freshness, Binding,. . .

Coalgebras

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Regular Behaviours with Names

State-finite

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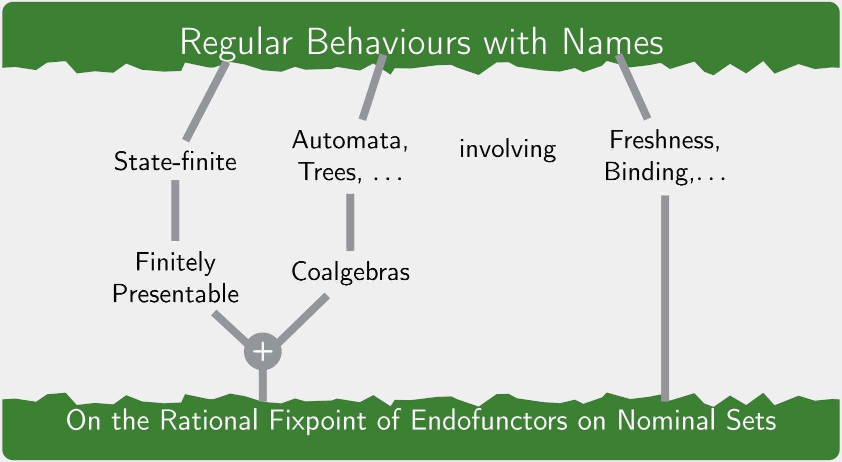
Regular Behaviours with Names
State-finite
Automata,
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involving
Freshness,
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Finitely
Presentable
Coalgebras
+
On the Rational Fixpoint of Endofunctors on Nominal Sets

Nom: The Category of Nominal Sets

Nominal Set X

Renaming atoms

A set X with

- Group action $\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times X \to X$

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λ-terms

$\mathrm{supp}(x) =$ all variables in x

λ-terms mod α-equivalence

$\mathrm{supp}(x) =$ **free** variables in x

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Equivariants $f : X \to Y$

Maps commuting with group actions: $\pi \cdot f(x) = f(\pi \cdot x)$.

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Behaviours with Names

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Gabbay, Pitts'99

$$LX = \mathcal{V} + \mathcal{V} \times X + X \times X \quad \overset{q_X}{\twoheadrightarrow} \quad L_\alpha X = \mathcal{V} + [\mathcal{V}]X + X \times X$$

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Final L-Coalgebra

λ-trees involving finitely many variables

$\not\twoheadrightarrow$

L_α-Coalgebras

λ-trees modulo α-equivalence involving finitely many free variables

Kurz, Petrisan, Severi, de Vries'13

The Rational Fixpoint: Regular Behaviours

$\mathcal{C}$ lfp category, F finitary

$$\text{f.p. } C \xrightarrow{c} FC$$

Adámek, Milius, Velebil'06

The Rational Fixpoint: Regular Behaviours

$\mathcal{C}$ lfp category, F finitary	$\mathcal{C} = \mathrm{Set}$	$FX = 2 \times X^A$
f.p. $C \xrightarrow{c} FC$	finite	finite automata

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$$\begin{array}{ccc} \text{f.p. } C & \xrightarrow{c} & FC \\ \downarrow & & \downarrow \\ D & \longrightarrow & FD \end{array}$$

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$D \longrightarrow FD$	locally finite	locally finite automata
$\varrho F \xrightarrow{r} F\varrho F$	final locally finite	Regular Languages

$$\begin{tikzcd}
\text{f.p. } C \arrow[r, "c"] \arrow[d] & FC \arrow[d] \\
D \arrow[r] \arrow[d, dashed, "!"'] & FD \arrow[d] \\
\varrho F \arrow[r, "r"] & F\varrho F
\end{tikzcd}$$

Rational Fixpoint

Adámek, Milius, Velebil'06

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Rational Fixpoint $= \operatorname*{colim}_{C \text{ f.p.}} C \xrightarrow{c} FC$

Adámek, Milius, Velebil'06

Finitely Presentable Objects in Nom

orbit-finite = finitely presentable in Nom

x, y in the same orbit $\Leftrightarrow \pi \cdot x = y$ for some π.

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Each orbit

Either singleton or **infinite**

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? Behaviour of $C \to FC$, C orbit-finite? ?

Regular Behaviours with Names

Rational λ-trees $= \varrho L$

$$LX = \mathcal{V} + \mathcal{V} \times X + X \times X$$

Rational λ-trees mod $\alpha = \varrho L_\alpha$

$$L_\alpha X = \mathcal{V} + [\mathcal{V}]X + X \times X$$

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? How to characterize rational fixpoints in Nom? ?

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$DX = 2 \times X^{\mathcal{V}}$, $KX = 2 \times X^{\mathcal{V}} \times [\mathcal{V}]X$,

Functors from Binding Signatures, …

Regular Behaviours with Names

$$\begin{array}{ccc} \mathsf{Nom} & \xrightarrow{\bar{F}} & \mathsf{Nom} \\ \mathsf{U}\downarrow & & \downarrow\mathsf{U} \\ \mathsf{Set} & \xrightarrow{F} & \mathsf{Set} \end{array}$$

Lifting of a Set-Functor

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Lifting of a Set-Functor

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$$G \overset{q}{\twoheadrightarrow} H$$

Quotient of another Nom-Functor

Rational λ-trees mod α $= \varrho L_\alpha$

$L_\alpha X = \mathcal{V} + [\mathcal{V}]X + X \times X$

? How to characterize rational fixpoints in Nom?
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Functors from Binding Signatures, ... ?

Part 1: Liftings

Liftings: Assumptions

Assumption: $\bar{F} : \mathsf{Nom} \circlearrowleft$ a localizable lifting, i.e.

1. $\bar{F}$ comes from a distributive law λ over $F = U\bar{F}D$.
2. For each $W \subseteq \mathcal{V}$, λ restricts to:
   $\lambda^W : \mathfrak{S}_\mathsf{f}(W) \times F\_ \to F(\mathfrak{S}_\mathsf{f}(W) \times \_)$.

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Examples

- Constants, Identity, Finitary Powerset $\mathcal{P}_\mathsf{f}$
- Closed under finite products, coproducts, composition.

Locally Finite vs Locally Orbit-Finite

$c : C \to FC$ in Set vs $c : C \to \bar{F}C$ in Nom

C finite $\not\Leftrightarrow$ C orbit-finite

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Lemma

For $c : C \to \bar{F}C$, c locally finite in Set $\Longrightarrow$ c is locally orbit-finite

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$(\varrho F, r)$ = Rational Fixpoint of $\bar{F} : \mathrm{Nom} \circlearrowleft$

Group action by corecursion. Finite support by coinducton.

Part 2: Quotients of Nom-functors

Quotients of coalgebras

$$G : \mathsf{Nom} \circlearrowleft \xrightarrow{q} H : \mathsf{Nom} \circlearrowleft$$

$$\varrho G \xrightarrow{r^g} G\varrho G \xrightarrow{q_{\varrho G}} H\varrho G$$

$$\varrho H \xrightarrow{r^H} H\varrho H$$

Quotients of coalgebras

$$G : \mathrm{Nom} \circlearrowleft \quad \overset{q}{\twoheadrightarrow} \quad H : \mathrm{Nom} \circlearrowleft$$

surjective?

$$\begin{tikzcd}
\varrho G \arrow[r, "r^g"] \arrow[d, dashed, "h"'] & G\varrho G \arrow[r, two heads, "q_{\varrho G}"] & H\varrho G \arrow[d, "Hh"] \\
\varrho H \arrow[rr, "r^H"] & & H\varrho H
\end{tikzcd}$$

Constructing backwards along the quotient

Definition

$$X < Y := \{(x, y) \in X \times Y \mid \mathrm{supp}(x) \subseteq \mathrm{supp}(y)\}$$

Sub-strength $s_{X,Y} : GX < Y \to G(X < Y)$,

not necessarily natural, but

$$\begin{tikzcd} GX < Y \arrow[rr, "s_{X,Y}"] \arrow[dr, "\mathrm{outl}"'] & & G(X < Y) \arrow[dl, "G\,\mathrm{outl}"] \\ & GX & \end{tikzcd}$$

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Examples

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Theorem

Finitary $G : \mathsf{Nom} \circlearrowleft$ with sub-strength & $q : G \twoheadrightarrow H$, then $\varrho G \twoheadrightarrow \varrho H$

Applications

$$F : \mathsf{Set}^{\circlearrowleft} \;\rightsquigarrow\; \bar{F} = G : \mathsf{Nom}^{\circlearrowleft} \;\twoheadrightarrow\; H : \mathsf{Nom}^{\circlearrowleft}$$

Example: Binding

$$LX = \mathcal{V} + \mathcal{V} \times X + X \times X \xrightarrow{q}\!\!\!\!\rightarrow L_\alpha X = \mathcal{V} + [\mathcal{V}]X + X \times X$$

Generally: L_α = Arbitrary Binding Signature, L = Polynomial

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Trees for the raw signature

- ϱL = rational λ-trees

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λ-trees modulo α-equivalence

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- νL = λ-trees with finitely many variables involved

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Example: Exponentiation by Atoms $X^{\mathcal{V}}$

default case exceptions

$$FX = \mathcal{V} \times X \times \coprod_{n\in\mathbb{N}} (\mathcal{V} \times X)^n \quad \overset{q}{\twoheadrightarrow} \quad HX = X^{\mathcal{V}}$$

Example: Exponentiation by Atoms $X^{\mathcal{V}}$

$$FX = \mathcal{V} \times \underbrace{X}_{\text{default case}} \times \coprod_{n \in \mathbb{N}} (\mathcal{V} \times \underbrace{X}_{\text{exceptions}})^n \overset{q}{\twoheadrightarrow} HX = X^{\mathcal{V}}$$

Various Kinds of Nominal Automata

- $FX = 2 \times X^{\mathcal{V}}$
- $KX = 2 \times X^{\mathcal{V}} \times [\mathcal{V}]X$
- $NX = 2 \times \mathcal{P}_{\mathrm{f}}(X^{\mathcal{V}}) \times \mathcal{P}_{\mathrm{f}}([\mathcal{V}]X)$

Open Questions

? More applications?
Necessity of asssumptions?
Any non-examples beside technical ones? ?

Main Results

Final Coalgebra: Infinite Behaviours	$\longleftarrow\!\!\!\not\!\!\!\longrightarrow$	Nominal Sets: finitely supported $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-action	$\longleftarrow\heartsuit\longrightarrow$	Rational Fixpoint: orbit-finite Behaviours

Main Results

Final Coalgebra: Infinite Behaviours $\longleftrightarrow$ Nominal Sets: finitely supported $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-action $\longleftrightarrow$ Rational Fixpoint: orbit-finite Behaviours

fails

- If $\bar{F} : \mathsf{Nom} \circlearrowleft$ is localizable lifting of $F : \mathsf{Set} \circlearrowleft$ then $\varrho\bar{F}$ is ϱF with corecursive $\mathfrak{S}_\mathsf{f}(\mathcal{V})$-action

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- If $G : \mathsf{Nom} \circlearrowleft$ is a quotient $H : \mathsf{Nom} \circlearrowleft$ with a substrength then ϱG is a quotient of ϱH

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Alexander Kurz, Daniela Petrisan, Jiri Velebil. "Algebraic Theories over Nominal Sets". In: **CoRR** abs/1006.3027 (2010).

Assumption 1: Comming from a Distributive Law

$\mathfrak{S}_{\mathsf{f}}(\mathcal{V})$-action on a Nominal Set X

Eilenberg-Moore algebra on X for the monad $TY = \mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times Y$

Liftings $\iff$ Distributive Laws

$$\begin{tikzcd} \mathsf{Set}^T \arrow[r, "\bar{F}"] \arrow[d, "U"'] & \mathsf{Set}^T \arrow[d, "U"] \\ \mathsf{Set} \arrow[r, "F"] & \mathsf{Set} \end{tikzcd} \quad \iff \quad \begin{array}{c} \lambda : TF \to FT \\ \text{preserving} \\ \text{monad structure} \end{array}$$

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$$

Assumption

$\bar{F}$: Nom $\to$ Nom comes from a Distributive Law $\mathfrak{S}_{\mathsf{f}}(\mathcal{V}) \times \_$ over F
Mapping nominal sets to nominal sets.

Assumption 2: Localizability

$\lambda : \mathfrak{S}_\mathsf{f}(\mathcal{V}) \times F\_ \to F(\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times \_)$ localizable

For each $W \subseteq \mathcal{V}$, λ restricts to $\mathfrak{S}_\mathsf{f}(\mathcal{V})$:

$$\begin{tikzcd}
\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times FX \arrow[r, "\lambda_X"] & F(\mathfrak{S}_\mathsf{f}(\mathcal{V}) \times X) \\
\mathfrak{S}_\mathsf{f}(W) \times FX \arrow[r, "\lambda_X^W"] \arrow[u, hook] & F(\mathfrak{S}_\mathsf{f}(W) \times X) \arrow[u, hook]
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\end{tikzcd}$$

Non-Example

For $F = \mathsf{Id}_{\mathsf{Set}}$, $\lambda(\pi, x) = (g \cdot \pi \cdot g^{-1})$, g fixed.

Something like “projective objects” in Nom

Definition: strongly supported

Some $x \in X$ is **strongly supported** iff

$$\pi \cdot x = x \Longrightarrow \forall v \in \mathrm{supp}(x) : \pi(v) = v$$

Examples

$\mathcal{V}^n$ is strongly supported. $\mathcal{P}_{\mathrm{f}}(\mathcal{V})$ not.

Proposition (Mentioned already in Kurz, Petrisan, Velebil’10)

X, Y nominal sets, X strongly supported, $O \subseteq X$ a choice of one element from each orbit. Then any map $f_0 : O \to Y$ with

$$\mathrm{supp}(f_0(x)) \subseteq \mathrm{supp}(x)$$

extends uniquely to an equivariant $f : X \to Y$.

Applied to our $C < W$

Lemma

There is a strong W and an equivariant map $f : C < W \to GC$ such that:

$$\begin{tikzcd}
C < W \arrow[r, "f"] \arrow[d, two heads, "\mathrm{outl}"'] & GC \arrow[d, two heads, "q_C"] \\
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$$\begin{tikzcd} C < W \arrow[r, "f"] \arrow[d, two heads, "\mathrm{outl}"'] & GC \arrow[d, two heads, "q_C"] \\ C \arrow[r, "c"] & HC \end{tikzcd}$$

Proposition

$c : C \to HC$ is via outl a quotient of the orbit-finite

$$C < W \xrightarrow{\bar{f}} GC < W \xrightarrow{s_{C,W}} G(C < W)$$

(where $\bar{f}(x, w) = (f(x), w)$).