Transitivity and Difunctionality of Bisimulations

Mehdi Zarrad and H. Peter Gumm

Marburg University

CMCS 2016, Eindhoven, the Netherlands

Bisimulations

Definition (Aczel,Mendler 1989)

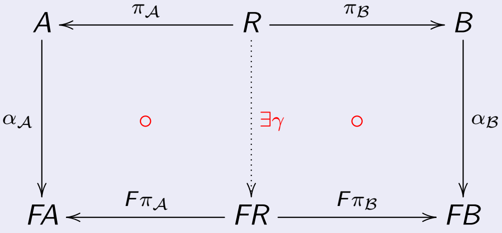
A *bisimulation* between coalgebras $\mathcal{A}$ and $\mathcal{B}$

$$\begin{array}{ccccc}
A & \xleftarrow{\pi_{\mathcal{A}}} & R & \xrightarrow{\pi_{\mathcal{B}}} & B \\
\Big\downarrow \alpha_{\mathcal{A}} & \circ & \vdots\, \exists\gamma & \circ & \Big\downarrow \alpha_{\mathcal{B}} \\
FA & \xleftarrow{F\pi_{\mathcal{A}}} & FR & \xrightarrow{F\pi_{\mathcal{B}}} & FB
\end{array}$$

Bisimulations

Definition (Aczel,Mendler 1989)

A *bisimulation* between coalgebras $\mathcal{A}$ and $\mathcal{B}$

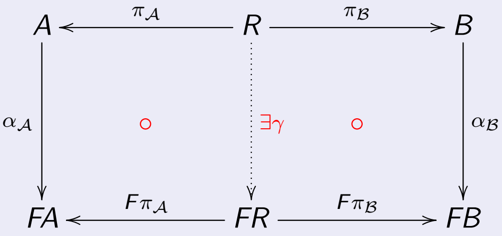


- Unions of bisimulations between $\mathcal{A}$ and $\mathcal{B}$ are bisimulation

Bisimulations

Definition (Aczel,Mendler 1989)

A *bisimulation* between coalgebras $\mathcal{A}$ and $\mathcal{B}$

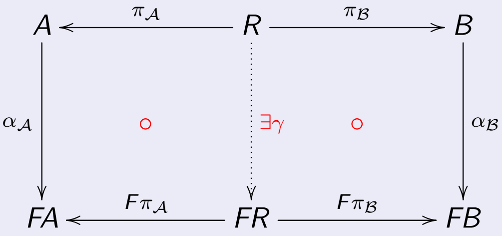


- Unions of bisimulations between $\mathcal{A}$ and $\mathcal{B}$ are bisimulation
  - The bisimulations between $\mathcal{A}$ and $\mathcal{B}$ form a complete lattice.

Bisimulations

Definition (Aczel,Mendler 1989)

A *bisimulation* between coalgebras $\mathcal{A}$ and $\mathcal{B}$

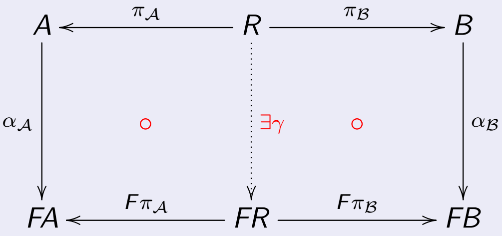


- Unions of bisimulations between $\mathcal{A}$ and $\mathcal{B}$ are bisimulation
  - The bisimulations between $\mathcal{A}$ and $\mathcal{B}$ form a complete lattice.
  - Denote by $[S]$ the union of all bisimulations contained in $S \subseteq A \times B$.

Bisimulations

Definition (Aczel,Mendler 1989)

A *bisimulation* between coalgebras $\mathcal{A}$ and $\mathcal{B}$



- Unions of bisimulations between $\mathcal{A}$ and $\mathcal{B}$ are bisimulation
  - The bisimulations between $\mathcal{A}$ and $\mathcal{B}$ form a complete lattice.
  - Denote by $[S]$ the union of all bisimulations contained in $S \subseteq A \times B$.
  - Denote the largest bisimulation $\sim_{\mathcal{A},\mathcal{B}} = [A \times B]$.

Congruences in Set_F

Definition

A *congruence* θ on a coalgebra $\mathcal{A}$ is the kernel of a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$,

Theorem (Gumm,T.Schröder 2005)

A congruence θ is regular iff $\theta = [\theta]^{\star}$.

Congruences in Set_F

Definition

A *congruence* θ on a coalgebra $\mathcal{A}$ is the kernel of a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$,

- θ is called regular, if φ regular epi.

Theorem (Gumm,T.Schröder 2005)

A congruence θ is regular iff $\theta = [\theta]^{\star}$.

Congruences in Set_F

Definition

A *congruence* θ on a coalgebra $\mathcal{A}$ is the kernel of a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$,

- θ is called regular, if φ regular epi.
- θ is called strictly regular, if θ is a bismulation.

Theorem (Gumm,T.Schröder 2005)

A congruence θ is regular iff $\theta = [\theta]^{\star}$.

Congruences in Set_F

Definition

A *congruence* θ on a coalgebra $\mathcal{A}$ is the kernel of a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$,

- θ is called regular, if φ regular epi.
- θ is called strictly regular, if θ is a bismulation.

- The set of all congruences on $\mathcal{A}$ forms a complete lattice $Cong(\mathcal{A})$.

Theorem (Gumm,T.Schröder 2005)

A congruence θ is regular iff $\theta = [\theta]^\star$.

Congruences in Set_F

Definition

A *congruence* θ on a coalgebra $\mathcal{A}$ is the kernel of a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$,

- θ is called regular, if φ regular epi.
- θ is called strictly regular, if θ is a bismulation.

- The set of all congruences on $\mathcal{A}$ forms a complete lattice $Cong(\mathcal{A})$.
- Denote the largest congruence by $\nabla_{\mathcal{A}}$.

Theorem (Gumm,T.Schröder 2005)

A congruence θ is regular iff $\theta = [\theta]^{\star}$.

Congruences in Set_F

Definition

A *congruence* θ on a coalgebra $\mathcal{A}$ is the kernel of a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$,

- θ is called regular, if φ regular epi.
- θ is called strictly regular, if θ is a bismulation.

- The set of all congruences on $\mathcal{A}$ forms a complete lattice *Cong*$(\mathcal{A})$.
- Denote the largest congruence by $\nabla_{\mathcal{A}}$.

Theorem (Gumm,T.Schröder 2005)

A congruence θ is regular iff $\theta = [\theta]^{\star}$.

Congruences in Set_F

Definition

A *congruence* θ on a coalgebra $\mathcal{A}$ is the kernel of a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$,

- θ is called regular, if φ regular epi.
- θ is called strictly regular, if θ is a bismulation.

- The set of all congruences on $\mathcal{A}$ forms a complete lattice *Cong*$(\mathcal{A})$.
- Denote the largest congruence by $\nabla_{\mathcal{A}}$.

Theorem (Gumm,T.Schröder 2005)

A congruence θ is regular iff $\theta = [\theta]^{\star}$.

- Every strictly regular congruence is regular.

2-congruences in Set_F

Definition

A $2-$*congruence* between two coalgebras $\mathcal{A}$ and $\mathcal{B}$ is the kernel (pullback in *the category Set*) of two homomorphisms $\varphi : \mathcal{A} \to \mathcal{C}$ and $\psi : \mathcal{B} \to \mathcal{C}$, i.e.

$$\theta = \ker(\varphi, \psi) := \{(a, b) \in A \times B \mid \varphi(a) = \psi(b)\}.$$

2-congruences in Set_F

Definition

A $2-$*congruence* between two coalgebras $\mathcal{A}$ and $\mathcal{B}$ is the kernel (pullback in *the category Set*) of two homomorphisms $\varphi : \mathcal{A} \to \mathcal{C}$ and $\psi : \mathcal{B} \to \mathcal{C}$, i.e.

$$\theta = \ker(\varphi, \psi) := \{(a, b) \in A \times B \mid \varphi(a) = \psi(b)\}.$$

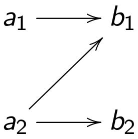
- *The largest $2-$congruence between $\mathcal{A}$ and $\mathcal{B}$ is called observational equivalence* and written $\nabla_{\mathcal{A},\mathcal{B}}$

Difunctionality

Definition

A relation $R \subseteq A \times B$ is called *difunctional (z-closed)* , if it satisfies :

$$(a_1, b_1), (a_2, b_1), (a_2, b_2) \in R \implies (a_1, b_2) \in R.$$

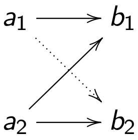


Difunctionality

Definition

A relation $R \subseteq A \times B$ is called *difunctional (z-closed)* , if it satisfies :

$$(a_1, b_1), (a_2, b_1), (a_2, b_2) \in R \implies (a_1, b_2) \in R.$$

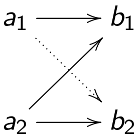


Difunctionality

Definition

A relation $R \subseteq A \times B$ is called *difunctional (z-closed)* , if it satisfies :

$$(a_1, b_1), (a_2, b_1), (a_2, b_2) \in R \implies (a_1, b_2) \in R.$$



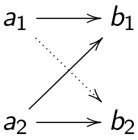
- A relation $R \subseteq A \times B$ is difunctional, if and only if it is a pullback of two maps in *Set*.

Difunctionality

Definition

A relation $R \subseteq A \times B$ is called *difunctional (z-closed)* , if it satisfies :

$$(a_1, b_1), (a_2, b_1), (a_2, b_2) \in R \implies (a_1, b_2) \in R.$$



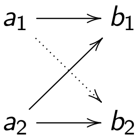
- A relation $R \subseteq A \times B$ is difunctional, if and only if it is a pullback of two maps in *Set*.
- The difunctional closure R^d can be obtained as the pullback of the pushout of π_A^R with π_B^R.

Difunctionality

Definition

A relation $R \subseteq A \times B$ is called *difunctional (z-closed)* , if it satisfies :

$$(a_1, b_1), (a_2, b_1), (a_2, b_2) \in R \implies (a_1, b_2) \in R.$$



- A relation $R \subseteq A \times B$ is difunctional, if and only if it is a pullback of two maps in *Set*.
- The difunctional closure R^d can be obtained as the pullback of the pushout of π_A^R with π_B^R.
- 2-congruences are difunctional.

Inclusions

Example

$2 \times \mathbb{P}_{\leq 3}-$coalgebra with $\sim_{\mathcal{A}} \neq \nabla_{\mathcal{A}}$

Inclusions

$$\sim_{\mathcal{A}} \subseteq \sim^*_{\mathcal{A}} \subseteq \nabla_{\mathcal{A}}$$

Example

$2 \times \mathbb{P}_{\leq 3}-$coalgebra with $\sim_{\mathcal{A}} \neq \nabla_{\mathcal{A}}$

Inclusions

$$\sim_{\mathcal{A}} \subseteq \sim^*_{\mathcal{A}} \subseteq \nabla_{\mathcal{A}}$$

$$\sim_{\mathcal{A},\mathcal{B}} \subseteq \sim^d_{\mathcal{A},\mathcal{B}} \subseteq \nabla_{\mathcal{A},\mathcal{B}}$$

Example

$2 \times \mathbb{P}_{\leq 3}-$coalgebra with $\sim_{\mathcal{A}} \neq \nabla_{\mathcal{A}}$

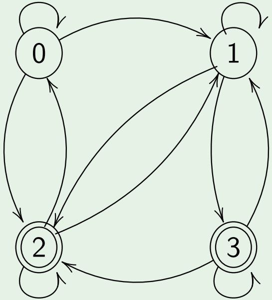
Inclusions

$$\sim_{\mathcal{A}} \subseteq \sim^*_{\mathcal{A}} \subseteq \nabla_{\mathcal{A}}$$

$$\sim_{\mathcal{A},\mathcal{B}} \subseteq \sim^d_{\mathcal{A},\mathcal{B}} \subseteq \nabla_{\mathcal{A},\mathcal{B}}$$

Example

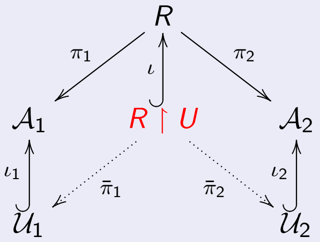
$2 \times \mathbb{P}_{\leq 3}-$coalgebra with $\sim_{\mathcal{A}} \neq \nabla_{\mathcal{A}}$



Restricting bisimulations

Definition

Let R a bisimulation between coalgebras $\mathcal{A}_1 = (A_1, \alpha_1)$ and $\mathcal{A}_2 = (A_2, \alpha_2)$, and $\mathcal{U}_i \leq \mathcal{A}_i$ subcoalgebras for $i = 1, 2$.
R restricts to $U := U_1 \times U_2$, if $R \restriction U := R \cap (U_1 \times U_2)$ is a bisimulation between $\mathcal{U}_1$ and $\mathcal{U}_2$.



Restricting bisimulations

Theorem (Gumm,T.Schröder 2005)

All bisimulations restrict to all subcoalgebras if and only if F preserves preimages.

Restricting bisimulations

Theorem (Gumm,T.Schröder 2005)

All bisimulations restrict to all subcoalgebras if and only if F preserves preimages.

Question

Which conditions on R guarantee that R restricts to $U_1 \times U_2$ without any condition on F?

Restricting bisimulations

Theorem

If there exist $\kappa_i : A_i \to U_i$ left inverses to the inclusion maps, satisfying for all $a_1 \in A_1$,$a_2 \in A_2$:

$$(a_1, a_2) \in R \implies (\kappa_1 a_1, \kappa_2 a_2) \in R$$

then R restricts to U.

Theorem

If $R[U_1] \subseteq U_2$ and $R^{-}[U_2] \subseteq U_1$ then R restricts to a bisimulation between $\mathcal{U}_1$ and $\mathcal{U}_2$.

Restricting bisimulations

Theorem

If there exist $\kappa_i : A_i \to U_i$ left inverses to the inclusion maps, satisfying for all $a_1 \in A_1$,$a_2 \in A_2$:

$$(a_1, a_2) \in R \implies (\kappa_1 a_1, \kappa_2 a_2) \in R$$

then R restricts to U.

The case where $R \restriction U$ is empty is trivial. Otherwise fix any pair $(u_1, u_2) \in R \restriction U$ and instantiate

$$\kappa_i(a) := \mathit{if}\ (a \in U_i)\ \mathit{then}\ a\ \mathit{else}\ u_i$$

we shall obtain

Theorem

If $R[U_1] \subseteq U_2$ and $R^-[U_2] \subseteq U_1$ then R restricts to a bisimulation between $\mathcal{U}_1$ and $\mathcal{U}_2$.

Restricting bisimulations

Theorem

If there exist $\kappa_i : A_i \to U_i$ left inverses to the inclusion maps, satisfying for all $a_1 \in A_1$,$a_2 \in A_2$:

$$(a_1, a_2) \in R \implies (\kappa_1 a_1, \kappa_2 a_2) \in R$$

then R restricts to U.

The case where $R \restriction U$ is empty is trivial. Otherwise fix any pair $(u_1, u_2) \in R \restriction U$ and instantiate

$$\kappa_i(a) := \mathit{if}\ (a \in U_i)\ \mathit{then}\ a\ \mathit{else}\ u_i$$

we shall obtain

Theorem

If $R[U_1] \subseteq U_2$ and $R^-[U_2] \subseteq U_1$ then R restricts to a bisimulation between $\mathcal{U}_1$ and $\mathcal{U}_2$.

Modification of coalgebra

Definition

Given a coalgebra $\mathcal{A} = (A, \alpha)$, element $x_0 \in A$ and subset $U \subseteq A$. We define a new coalgebra $\mathcal{A}_{x_0}^{U} := (A, \bar{\alpha})$

$$\bar{\alpha}(x) := \begin{cases} \alpha(x_0) & \textit{if } x \in U \\ \alpha(x) & \textit{else}. \end{cases}$$

Modification of coalgebra

Definition

Given a coalgebra $\mathcal{A} = (A, \alpha)$, element $x_0 \in A$ and subset $U \subseteq A$. We define a new coalgebra $\mathcal{A}^U_{x_0} := (A, \bar{\alpha})$

$$\bar{\alpha}(x) := \begin{cases} \alpha(x_0) & \textit{if } x \in U \\ \alpha(x) & \textit{else}. \end{cases}$$

Preservation of homomorphisms and congruences

Let $\varphi : \mathcal{A} \to \mathcal{C}$ be a homomorphism and $U \subseteq [x_0] \ker \varphi$ for some $x_0 \in A$. Then the map $\varphi : A \to C$ is also a homomorphism $\varphi : \bar{\mathcal{A}} \to \mathcal{C}$ where $\bar{\mathcal{A}} := \mathcal{A}^U_{x_0}$.

Modification of coalgebra

Definition

Given a coalgebra $\mathcal{A} = (A, \alpha)$, element $x_0 \in A$ and subset $U \subseteq A$. We define a new coalgebra $\mathcal{A}_{x_0}^U := (A, \bar{\alpha})$

$$\bar{\alpha}(x) := \begin{cases} \alpha(x_0) & \textit{if } x \in U \\ \alpha(x) & \textit{else}. \end{cases}$$

Preservation of homomorphisms and congruences

Let $\varphi : \mathcal{A} \to \mathcal{C}$ be a homomorphism and $U \subseteq [x_0] \ker \varphi$ for some $x_0 \in A$. Then the map $\varphi : A \to C$ is also a homomorphism $\varphi : \bar{\mathcal{A}} \to \mathcal{C}$ where $\bar{\mathcal{A}} := \mathcal{A}_{x_0}^U$.
If $U \subseteq [x_0]\nabla_{\mathcal{A}}$ and $V \subseteq [y_0]\nabla_{\mathcal{B}}$ then $\nabla_{\mathcal{A},\mathcal{B}} = \nabla_{\bar{\mathcal{A}},\bar{\mathcal{B}}}$.

Weak preservation of kernel pairs

Theorem (Gumm,T.Schröder 2005)

F weakly preserves kernel pairs iff every congruence is strictly regular.

Theorem

F weakly preserves kernel pairs iff every congruence is regular.

Weak preservation of kernel pairs

Theorem (Gumm,T.Schröder 2005)

F weakly preserves kernel pairs iff every congruence is strictly regular.

Theorem

F weakly preserves kernel pairs iff every congruence is regular.

Weak preservation of kernel pairs

Theorem

The following are equivalent:

1. *F preserves weak kernel pairs,*

Weak preservation of kernel pairs

Theorem

The following are equivalent:

1. *F preserves weak kernel pairs,*
2. *R^d is a bisimulation, whenever R is,*

Weak preservation of kernel pairs

Theorem

The following are equivalent:

1. *F preserves weak kernel pairs,*
2. *R^d is a bisimulation, whenever R is,*
3. *$[\theta]$ is difunctional for each $2-$congruence θ,*

Weak preservation of kernel pairs

Theorem

The following are equivalent:

1. *F preserves weak kernel pairs,*
2. *R^d is a bisimulation, whenever R is,*
3. *$[\theta]$ is difunctional for each $2-$congruence θ,*
4. *$[\theta]$ is transitive for each congruence θ,*

Weak preservation of kernel pairs

Theorem

The following are equivalent:

1. F *preserves weak kernel pairs,*
2. R^d *is a bisimulation, whenever* R *is,*
3. $[\theta]$ *is difunctional for each* $2-$*congruence* θ,
4. $[\theta]$ *is transitive for each congruence* θ,
5. $[\theta]^* = \theta$ *for every congruence* θ,

Weak preservation of kernel pairs

Theorem

The following are equivalent:

1. *F preserves weak kernel pairs,*
2. *R^d is a bisimulation, whenever R is,*
3. *$[\theta]$ is difunctional for each $2-$congruence θ,*
4. *$[\theta]$ is transitive for each congruence θ,*
5. *$[\theta]^* = \theta$ for every congruence θ,*
6. *The regular congruences form a sublattice of $Cong(\mathcal{A})$.*

Transitivity of $\sim$

Theorem (Gumm, T.Schröder 2005)

If F preserves preimages, then the following are equivalent:

1. $\sim_{\mathcal{A}} = \nabla_{\mathcal{A}}$ *for each* $F-$*coalgebra* $\mathcal{A}$.
2. $\sim_{\mathcal{A}}$ *is transitive for each* $F-$*coalgebra* $\mathcal{A}$.

Transitivity of $\sim$

Theorem (Gumm,T.Schröder 2005)

If F preserves preimages, then the following are equivalent:

1. $\sim_{\mathcal{A}} = \nabla_{\mathcal{A}}$ *for each* $F-$*coalgebra* $\mathcal{A}$.
2. $\sim_{\mathcal{A}}$ *is transitive for each* $F-$*coalgebra* $\mathcal{A}$.

Idea

We attempt to use the restricting bisimulation without assumption of preimage preservation.

Transitivity of $\sim$

Theorem

The following are equivalent:

1. $\sim_{\mathcal{A}} = \nabla_{\mathcal{A}}$ *for each* $F-$*coalgebra* $\mathcal{A}$.

Transitivity of $\sim$

Theorem

The following are equivalent:

1. $\sim_{\mathcal{A}} = \nabla_{\mathcal{A}}$ *for each* $F-$*coalgebra* $\mathcal{A}$.
2. $\sim_{\mathcal{A}}$ *is transitive for each* $F-$*coalgebra* $\mathcal{A}$.

Transitivity of $\sim$

Theorem

The following are equivalent:

1. $\sim_{\mathcal{A}} = \nabla_{\mathcal{A}}$ *for each* $F-$*coalgebra* $\mathcal{A}$.
2. $\sim_{\mathcal{A}}$ *is transitive for each* $F-$*coalgebra* $\mathcal{A}$.
3. $\sim^*_{\mathcal{A}} = \nabla_{\mathcal{A}}$ *for each* $F-$*coalgebra* $\mathcal{A}$.,

Transitivity of $\sim$

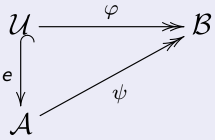
Theorem

The following are equivalent:

1. $\sim_{\mathcal{A}} = \nabla_{\mathcal{A}}$ *for each* $F-$*coalgebra* $\mathcal{A}$.
2. $\sim_{\mathcal{A}}$ *is transitive for each* $F-$*coalgebra* $\mathcal{A}$.
3. $\sim_{\mathcal{A}}^{*} = \nabla_{\mathcal{A}}$ *for each* $F-$*coalgebra* $\mathcal{A}$.,

Proof-steps (2)⇒(1)

Lemma: If an epimorphism $\varphi : \mathcal{U} \twoheadrightarrow \mathcal{B}$ can be factored as $\varphi = \psi \circ e$ with $e : \mathcal{U} \hookrightarrow \mathcal{A}$ regular mono and $\psi : \mathcal{A} \twoheadrightarrow \mathcal{B}$ a strictly regular epi, then φ is strictly regular epi.



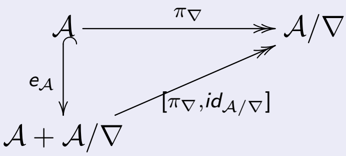
Transitivity of $\sim$

Theorem

The following are equivalent:

1. $\sim_{\mathcal{A}} = \nabla_{\mathcal{A}}$ *for each* $F-$*coalgebra* $\mathcal{A}$.
2. $\sim_{\mathcal{A}}$ *is transitive for each* $F-$*coalgebra* $\mathcal{A}$.
3. $\sim^{*}_{\mathcal{A}} = \nabla_{\mathcal{A}}$ *for each* $F-$*coalgebra* $\mathcal{A}$.,

Proof-steps (2)⇒(1)



Transitivity of $\sim$

Theorem

The following are equivalent:

1. $\sim_{\mathcal{A}} = \nabla_{\mathcal{A}}$ *for each* $F-$*coalgebra* $\mathcal{A}$.
2. $\sim_{\mathcal{A}}$ *is transitive for each* $F-$*coalgebra* $\mathcal{A}$.
3. $\sim^{*}_{\mathcal{A}} = \nabla_{\mathcal{A}}$ *for each* $F-$*coalgebra* $\mathcal{A}$.,

Transitivity of $\sim$

Theorem

The following are equivalent:

1. $\sim_{\mathcal{A}} = \nabla_{\mathcal{A}}$ *for each* $F-$*coalgebra* $\mathcal{A}$.
2. $\sim_{\mathcal{A}}$ *is transitive for each* $F-$*coalgebra* $\mathcal{A}$.
3. $\sim^*_{\mathcal{A}} = \nabla_{\mathcal{A}}$ *for each* $F-$*coalgebra* $\mathcal{A}$.,

Question

Generalization to relations between two coalgebras?

Transitivity of $\sim$

Theorem

The following are equivalent:

1. $\sim_{\mathcal{A}} = \nabla_{\mathcal{A}}$ *for each* $F-$*coalgebra* $\mathcal{A}$.
2. $\sim_{\mathcal{A}}$ *is transitive for each* $F-$*coalgebra* $\mathcal{A}$.
3. $\sim^{*}_{\mathcal{A}} = \nabla_{\mathcal{A}}$ *for each* $F-$*coalgebra* $\mathcal{A}$.,

Question

Generalization to relations between two coalgebras?

Idea

We choose difunctionality as generalization of transitivity

Difunctionality of $\sim$

We consider the following conditions:

1. For all $F-$coalgebras $\mathcal{A}, \mathcal{B}$: $\sim_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$
2. For all $F-$coalgebras $\mathcal{A}, \mathcal{B}$: $\sim_{\mathcal{A},\mathcal{B}}$ is difunctional.
3. For all $F-$coalgebras $\mathcal{A}, \mathcal{B}$: $\sim^{d}_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$

Difunctionality of $\sim$

Theorem

If F preserves preimages, then the following are equivalent:

1. *For all* $F-$*coalgebras* $\mathcal{A}, \mathcal{B}$: $\sim_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$
2. *For all* $F-$*coalgebras* $\mathcal{A}, \mathcal{B}$: $\sim_{\mathcal{A},\mathcal{B}}$ *is difunctional.*

Bisimulation as compatible relation

In universal algebra compatible relation R satisfies

$$(x_1, ..., x_n)\, R^n\, (y_1, ..., y_n) \implies f\,(x_1, ..., x_n)\, R\, f\,(y_1, ..., y_n)$$

Bisimulation as compatible relation

In universal algebra compatible relation R satisfies

$$(x_1, ..., x_n)\, R^n\, (y_1, ..., y_n) \implies f\,(x_1, ..., x_n)\, R\, f\,(y_1, ..., y_n)$$

In the case universal coalgebra

Bisimulation as compatible relation

In universal algebra compatible relation R satisfies

$$(x_1, ..., x_n)\, R^n\, (y_1, ..., y_n) \implies f\,(x_1, ..., x_n)\, R\, f\,(y_1, ..., y_n)$$

In the case universal coalgebra

$$x\, R\, y \implies \alpha(x)\, L(R)\, \alpha(y)$$

for a "lifting" of R to a relation $L(R) \subseteq FA \times FB$

Relation Lifting

Definition

A relation lifting L is a transformation sending every relation $R \subseteq A \times B$ into a relation $L(R) \subseteq FA \times FB$.

Relation Lifting

Definition

A relation lifting L is a transformation sending every relation $R \subseteq A \times B$ into a relation $L(R) \subseteq FA \times FB$.
It is called monotonic, if $R \subseteq S$ implies $L(R) \subseteq L(S)$.

Relation Lifting

Definition

A relation lifting L is a transformation sending every relation $R \subseteq A \times B$ into a relation $L(R) \subseteq FA \times FB$.
It is called monotonic, if $R \subseteq S$ implies $L(R) \subseteq L(S)$.

- The Barr extension $\bar{F}$ where $\bar{F}$ is the relation lifting

$$\bar{F}(R) := \{(F\pi_A^R(u), F\pi_B^R(u)) \mid u \in F(R)\}.$$

is an example of a monotonic relation lifting.

L-simulation

Definition

$L-$simulation is a relation $R \subseteq A \times B$ such

$$x \, R \, y \implies \alpha_{\mathcal{A}}(x) \, L(R) \, \alpha_{\mathcal{B}}(y).$$

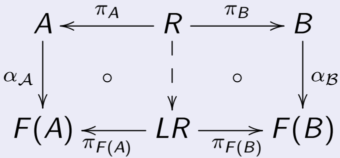
L-simulation

Definition

$L-$simulation is a relation $R \subseteq A \times B$ such

$$x \, R \, y \implies \alpha_{\mathcal{A}}(x) \, L(R) \, \alpha_{\mathcal{B}}(y).$$

Diagrammatically



L-simulation

Definition

$L-$simulation is a relation $R \subseteq A \times B$ such

$$x\,R\,y \implies \alpha_{\mathcal{A}}(x)\,L(R)\,\alpha_{\mathcal{B}}(y).$$

Diagrammatically

$$\begin{array}{ccccc}
A & \xleftarrow{\pi_A} & R & \xrightarrow{\pi_B} & B \\
\downarrow{\scriptstyle \alpha_{\mathcal{A}}} & \circ & \downarrow & \circ & \downarrow{\scriptstyle \alpha_{\mathcal{B}}} \\
F(A) & \xleftarrow[\pi_{F(A)}]{} & LR & \xrightarrow[\pi_{F(B)}]{} & F(B)
\end{array}$$

- Denote the union of all $L-$simulations between $\mathcal{A}$ and $\mathcal{B}$ by $\approx^{L}_{\mathcal{A},\mathcal{B}}$.

L-simulation

Definition

$L-$simulation is a relation $R \subseteq A \times B$ such

$$x\,R\,y \implies \alpha_{\mathcal{A}}(x)\,L(R)\,\alpha_{\mathcal{B}}(y).$$

Diagrammatically

$$\begin{array}{ccccc}
A & \xleftarrow{\pi_A} & R & \xrightarrow{\pi_B} & B \\
{\scriptstyle\alpha_{\mathcal{A}}}\downarrow & \circ & \downarrow & \circ & \downarrow{\scriptstyle\alpha_{\mathcal{B}}} \\
F(A) & \xleftarrow[\pi_{F(A)}]{} & LR & \xrightarrow[\pi_{F(B)}]{} & F(B)
\end{array}$$

- Denote the union of all $L-$simulations between $\mathcal{A}$ and $\mathcal{B}$ by $\approx^{L}_{\mathcal{A},\mathcal{B}}$.
- If L is monotonic then $\approx^{L}_{\mathcal{A},\mathcal{B}}$ is again an $L-$simulation.

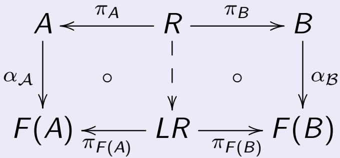
L-simulation

Definition

$L-$simulation is a relation $R \subseteq A \times B$ such

$$x\,R\,y \implies \alpha_{\mathcal{A}}(x)\,L(R)\,\alpha_{\mathcal{B}}(y).$$

Diagrammatically



- Denote the union of all $L-$simulations between $\mathcal{A}$ and $\mathcal{B}$ by $\approx^{L}_{\mathcal{A},\mathcal{B}}$.
- If L is monotonic then $\approx^{L}_{\mathcal{A},\mathcal{B}}$ is again an $L-$simulation.
- For $L = \bar{F}$: Alternative definition of bisimulation (Hermida,Jacobs 1998) and $\approx^{L}_{\mathcal{A},\mathcal{B}}$ agrees with $\sim_{\mathcal{A},\mathcal{B}}$.

Counterexample (2)⇒(1)

1. For all $F-$coalgebras $\mathcal{A}, \mathcal{B}$: $\sim_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$
2. For all $F-$coalgebras $\mathcal{A}, \mathcal{B}$: $\sim_{\mathcal{A},\mathcal{B}}$ is difunctional

Example

The neighborhood functor 2^{2^-}.

Counterexample (2)⇒(1)

1. For all $F-$coalgebras $\mathcal{A}, \mathcal{B}$: $\sim_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$
2. For all $F-$coalgebras $\mathcal{A}, \mathcal{B}$: $\sim_{\mathcal{A},\mathcal{B}}$ is difunctional

Example

The neighborhood functor 2^{2^-}.

- $\sim_{\mathcal{A},\mathcal{B}}$ is difunctional for all coalgebras $\mathcal{A}, \mathcal{B}$, since 2^{2^-} weakly preserves kernel pairs.

Counterexample (2)⇒(1)

1. For all $F-$coalgebras $\mathcal{A},\mathcal{B}$: $\sim_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$
2. For all $F-$coalgebras $\mathcal{A},\mathcal{B}$: $\sim_{\mathcal{A},\mathcal{B}}$ is difunctional

Example

The neighborhood functor 2^{2^-}.

- $\sim_{\mathcal{A},\mathcal{B}}$ is difunctional for all coalgebras $\mathcal{A},\mathcal{B}$, since 2^{2^-} weakly preserves kernel pairs.
- But $\sim_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$ does not hold, because there is no relation lifting L for the neighborhood functor in the sense that $\approx^L_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$ for all coalgebras $\mathcal{A},\mathcal{B}$ (Marti,Venema 2012)

Extensions of L-simulations

Definition

A relation lifting L is *extensible*, if for all coalgebras $\mathcal{A}$ and $\mathcal{B}$
If $R \subseteq \mathcal{A} \times \mathcal{B}$ is a $L-$simulation then so is each Relation R' between R and $ker\,\alpha_{\mathcal{A}} \circ R \circ ker\,\alpha_{\mathcal{B}}$.

Extensions of L-simulations

Definition

A relation lifting L is *extensible*, if for all coalgebras $\mathcal{A}$ and $\mathcal{B}$
If $R \subseteq \mathcal{A} \times \mathcal{B}$ is a $L-$simulation then so is each Relation R' between R and $ker\, \alpha_{\mathcal{A}} \circ R \circ ker\, \alpha_{\mathcal{B}}$.

Proposition

A relation lifting L is monotonic iff it is extensible.

Monotonic relation Lifting

Theorem

For a monotonic relation lifting L, the following are equivalent:

1. *For all coalgebras* $\mathcal{A}, \mathcal{B}$*:* $\approx_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$
2. *For all coalgebras* $\mathcal{A}, \mathcal{B}$*:* $\approx^{d}_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$

Proof-steps (2)⇒(1)

Monotonic relation Lifting

Theorem

For a monotonic relation lifting L, the following are equivalent:

1. *For all coalgebras* $\mathcal{A}, \mathcal{B}$*:* $\approx_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$
2. *For all coalgebras* $\mathcal{A}, \mathcal{B}$*:* $\approx^{d}_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$

Proof-steps (2)⇒(1)

Monotonic relation Lifting

Theorem

For a monotonic relation lifting L, the following are equivalent:

1. *For all coalgebras* $\mathcal{A}, \mathcal{B}$: $\approx_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$
2. *For all coalgebras* $\mathcal{A}, \mathcal{B}$: $\approx^{d}_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$

Proof-steps (2)⇒(1)

We have to show that $\nabla_{\mathcal{A},\mathcal{B}}$ is an $L-$simulation. Given $(x_0, y_0) \in \nabla_{\mathcal{A},\mathcal{B}}$.

Monotonic relation Lifting

Theorem

For a monotonic relation lifting L, the following are equivalent:

1. *For all coalgebras $\mathcal{A}, \mathcal{B}$: $\approx_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$*
2. *For all coalgebras $\mathcal{A}, \mathcal{B}$: $\approx^{d}_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$*

Proof-steps (2)⇒(1)

We have to show that $\nabla_{\mathcal{A},\mathcal{B}}$ is an $L-$simulation. Given $(x_0, y_0) \in \nabla_{\mathcal{A},\mathcal{B}}$. Define $\bar{\mathcal{A}} := \mathcal{A}^U_{x_0}$ and $\bar{\mathcal{B}} := \mathcal{B}^V_{y_0}$ where $U := [x_0]\nabla_{\mathcal{A}}$ and $V := [y_0]\nabla_{\mathcal{B}}$.

Monotonic relation Lifting

Theorem

For a monotonic relation lifting L, the following are equivalent:

1. *For all coalgebras $\mathcal{A}, \mathcal{B}$: $\approx_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$*
2. *For all coalgebras $\mathcal{A}, \mathcal{B}$: $\approx^d_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$*

Proof-steps (2)⇒(1)

We have to show that $\nabla_{\mathcal{A},\mathcal{B}}$ is an $L-$simulation. Given $(x_0, y_0) \in \nabla_{\mathcal{A},\mathcal{B}}$.
Define $\bar{\mathcal{A}} := \mathcal{A}^U_{x_0}$ and $\bar{\mathcal{B}} := \mathcal{B}^V_{y_0}$ where $U := [x_0]\nabla_{\mathcal{A}}$ and $V := [y_0]\nabla_{\mathcal{B}}$.
$\nabla_{\bar{\mathcal{A}},\bar{\mathcal{B}}} = \nabla_{\mathcal{A},\mathcal{B}}$

Monotonic relation Lifting

Theorem

For a monotonic relation lifting L, the following are equivalent:

1. *For all coalgebras $\mathcal{A}, \mathcal{B}$: $\approx_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$*
2. *For all coalgebras $\mathcal{A}, \mathcal{B}$: $\approx^d_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$*

Proof-steps (2)⇒(1)

We have to show that $\nabla_{\mathcal{A},\mathcal{B}}$ is an $L-$simulation. Given $(x_0, y_0) \in \nabla_{\mathcal{A},\mathcal{B}}$.
Define $\bar{\mathcal{A}} := \mathcal{A}^U_{x_0}$ and $\bar{\mathcal{B}} := \mathcal{B}^V_{y_0}$ where $U := [x_0]\nabla_{\mathcal{A}}$ and $V := [y_0]\nabla_{\mathcal{B}}$.
$\nabla_{\bar{\mathcal{A}},\bar{\mathcal{B}}} = \nabla_{\mathcal{A},\mathcal{B}}$
By hypothesis $(x_0, y_0) \in \approx^d_{\bar{\mathcal{A}},\bar{\mathcal{B}}}$, so there is $x_1 \in A$ with $x_1 \approx_{\bar{\mathcal{A}},\bar{\mathcal{B}}} y_0$.

Monotonic relation Lifting

Theorem

For a monotonic relation lifting L, the following are equivalent:

1. *For all coalgebras* $\mathcal{A}, \mathcal{B}$*:* $\approx_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$
2. *For all coalgebras* $\mathcal{A}, \mathcal{B}$*:* $\approx^{d}_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$

Proof-steps (2)⇒(1)

We have to show that $\nabla_{\mathcal{A},\mathcal{B}}$ is an $L-$simulation. Given $(x_0, y_0) \in \nabla_{\mathcal{A},\mathcal{B}}$.
Define $\bar{\mathcal{A}} := \mathcal{A}^{U}_{x_0}$ and $\bar{\mathcal{B}} := \mathcal{B}^{V}_{y_0}$ where $U := [x_0]\nabla_{\mathcal{A}}$ and $V := [y_0]\nabla_{\mathcal{B}}$.
$\nabla_{\bar{\mathcal{A}},\bar{\mathcal{B}}} = \nabla_{\mathcal{A},\mathcal{B}}$
By hypothesis $(x_0, y_0) \in \approx^{d}_{\bar{\mathcal{A}},\bar{\mathcal{B}}}$, so there is $x_1 \in A$ with $x_1 \approx_{\bar{\mathcal{A}},\bar{\mathcal{B}}} y_0$.
Using extensibility we obtain $x_0 \approx_{\bar{\mathcal{A}},\bar{\mathcal{B}}} y_0$

Monotonic relation Lifting

Theorem

For a monotonic relation lifting L, the following are equivalent:

1. *For all coalgebras $\mathcal{A}, \mathcal{B}$: $\approx_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$*
2. *For all coalgebras $\mathcal{A}, \mathcal{B}$: $\approx^{d}_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$*

Proof-steps (2)⇒(1)

We have to show that $\nabla_{\mathcal{A},\mathcal{B}}$ is an $L-$simulation. Given $(x_0, y_0) \in \nabla_{\mathcal{A},\mathcal{B}}$.
Define $\bar{\mathcal{A}} := \mathcal{A}^{U}_{x_0}$ and $\bar{\mathcal{B}} := \mathcal{B}^{V}_{y_0}$ where $U := [x_0]\nabla_{\mathcal{A}}$ and $V := [y_0]\nabla_{\mathcal{B}}$.
$\nabla_{\bar{\mathcal{A}},\bar{\mathcal{B}}} = \nabla_{\mathcal{A},\mathcal{B}}$
By hypothesis $(x_0, y_0) \in \approx^{d}_{\bar{\mathcal{A}},\bar{\mathcal{B}}}$, so there is $x_1 \in A$ with $x_1 \approx_{\bar{\mathcal{A}},\bar{\mathcal{B}}} y_0$.
Using extensibility we obtain $x_0 \approx_{\bar{\mathcal{A}},\bar{\mathcal{B}}} y_0$
Finally $(\alpha(x_0), \beta(y_0)) \in L(\approx_{\bar{\mathcal{A}},\bar{\mathcal{B}}}) \subseteq L(\nabla_{\bar{\mathcal{A}},\bar{\mathcal{B}}}) = L(\nabla_{\mathcal{A},\mathcal{B}})$

Monotonic relation Lifting

Choose $L := \bar{F}$

Corollary

The following are equivalent:

1. *For all coalgebras* $\mathcal{A}, \mathcal{B}$: $\sim_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$.
2. *For all coalgebras* $\mathcal{A}, \mathcal{B}$: $\sim^{d}_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$.

Monotonic relation Lifting

Choose $L := \bar{F}$

Corollary

The following are equivalent:

1. *For all coalgebras* $\mathcal{A}, \mathcal{B}$*:* $\sim_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$.
2. *For all coalgebras* $\mathcal{A}, \mathcal{B}$*:* $\sim^{d}_{\mathcal{A},\mathcal{B}} = \nabla_{\mathcal{A},\mathcal{B}}$.

Conclusion

We have found conditions under which:

- Bisimulations restrict to subcoalgebras.
- Observational equivalence and bisimilarity agree
  - on single coalgebras
  - on pairs of coalgebras
- Transitivity has to be replaced by difunctionality.

open

A characterization in Set_F:

$$\begin{aligned} & F \text{ weakly preserves kernel pairs} \\ \Leftrightarrow\ & \text{every epi is regular} \\ \Rightarrow\ & \text{every mono is regular} \end{aligned}$$

Take home question: $\Leftarrow$?