Relating Structure to Power: Comonadic semantics for computational resources

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Structure *vs* Power: The Great Divide

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- compositionality, semantics
- How we can master the complexity of computer systems and software?

Power:

- expressiveness, complexity
- How we can harness the power of computation and recognize its limits?

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Mazur quoting Lenstra:

> *twenty years ago he was firm in his conviction that he DID want to solve Diophantine equations, and that he DID NOT wish to represent functors – and now he is amused to discover himself representing functors in order to solve Diophantine equations!*

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- "The Pebbling Comonad in Finite Model Theory", SA, Anuj Dawar and Pengming Wang, LiCS 2017.
- "Relating Structure to Power: comonadic semantics for computational resources", SA and Nihil Shah, extended abstract in CMCS proceedings, conference version submitted.

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Our setting will be $\mathcal{R}(\sigma)$, the category of relational structures and homomorphisms.

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- In most cases of interest in FMT, the converse is true too.
- In descriptive complexity, we seek to characterize a complexity class **C** (for decision problems) as those classes of structures $\mathcal{K}$ (e.g. graphs) definable in $\mathcal{L}$.

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The EF-game between $\mathcal{A}$ and $\mathcal{B}$. In the i'th round, Spoiler moves by choosing an element in A or B; Duplicator responds by choosing an element in the other structure. Duplicator wins after k rounds if the relation $\{(a_i, b_i) \mid 1 \leq i \leq k\}$ is a partial isomorphism.

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Similarly, there are k-pebble games, and bismulation games payed to depth k.

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- Thus the notion of local approximation built into the game is internalised into the category of σ-structures and homomorphisms.
- This leads to comonadic and coalgebraic characterisations of a number of central concepts in Finite Model Theory and combinatorics.

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This is easily verified to yield a comonad on $\mathcal{R}(\sigma)$.

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Theorem

The following are equivalent:

1. *There is a homomorphism* $\mathbb{E}_k\mathcal{A} \to \mathcal{B}$.
2. *Duplicator has a winning strategy for the existential Ehrenfeucht-Fraïssé game with* k *rounds, played from* $\mathcal{A}$ *to* $\mathcal{B}$.
3. *For every existential positive sentence* φ *with quantifier rank* $\leq k$, $\mathcal{A} \models \varphi \;\Rightarrow\; \mathcal{B} \models \varphi$.

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We define $\mathcal{A} \leftrightarrow_k^{\mathbb{E}} \mathcal{B}$ iff there are non-empty sets $F \subseteq S$, $G \subseteq T$, which are *locally invertible* in the following sense:

1. For all $f \in F$, $s \in A^{\leq k}$, for some $g \in G$, $g^* f^*(s) = s$.
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Proposition

The following are equivalent:

1. $\mathcal{A} \leftrightarrow_k^{\mathbb{E}} \mathcal{B}$.
2. *There is a winning strategy for Duplicator in the k-round Ehrenfeucht-Fraïssé game between $\mathcal{A}$ and $\mathcal{B}$.*

A fixpoint characterization

Define set functions $\Gamma : \mathcal{P}(S) \to \mathcal{P}(T)$, $\Delta : \mathcal{P}(T) \to \mathcal{P}(S)$:

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If $\mathcal{A}$ and $\mathcal{B}$ are finite, so is S, and we can construct the greatest fixpoint by a finite descending sequence

$$S \supseteq \Theta(S) \supseteq \Theta^2(S) \supseteq \cdots$$

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We can generically define two equivalences based on our indexed comonads $\mathbb{E}_k$:

- $\mathcal{A} \rightleftarrows_k^{\mathbb{E}} \mathcal{B}$ iff there are coKleisli morphisms $\mathbb{E}_k\mathcal{A} \to \mathcal{B}$ and $\mathbb{E}_k\mathcal{B} \to \mathcal{A}$. Note that there need be no relationship between these morphisms.
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Just as for EF-games, there is an existential-positive version, in which Spoiler only plays in $\mathcal{A}$, and Duplicator responds in $\mathcal{B}$.

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Given a homomorphism $f : \mathbb{P}_k\mathcal{A} \to \mathcal{B}$, we define the coextension $f^* : \mathbb{P}_k\mathcal{A} \to \mathbb{P}_k\mathcal{B}$ by

$$f^*[(p_1, a_1), \ldots, (p_j, a_j)] = [(p_1, b_1), \ldots, (p_j, b_j)],$$

where $b_i = f[(p_1, a_1), \ldots, (p_i, a_i)]$, $1 \leq i \leq j$.

Logical equivalences

Again, we will be interested in three logics in relation to the pebbling comonad:

- One is $\mathcal{L}^k$, the k-variable fragment of (infinitary) first-order logic.
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Interestingly, the intermediate equivalence $\mathcal{A} \leftrightarrow^{\mathbb{P}}_k \mathcal{B}$ (back-and-forth without isomorphism) can be defined from just a *single* pair of morphisms $f : \mathbb{P}_k\mathcal{A} \to \mathcal{B}$ and $g : \mathbb{P}_k\mathcal{B} \to \mathcal{A}$, satisfying a certain "compatibility" relation.

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The universe of $\mathbb{M}_k(\mathcal{A}, a)$ comprises $[a]$, which is the distinguished element, together with all sequences of the form $[a_0, \alpha_1, a_1, \ldots, \alpha_j, a_j]$, where $a = a_0$, $1 \leq j \leq k$, and $R^{\mathcal{A}}_{\alpha_i}(a_i, a_{i+1})$, $0 \leq i < j$.

Simulation and Bisimulation

The resource index of $\mathbb{M}_k$ corresponds to the *level of approximation* in simulation $\preceq_k$ and bisimulation $\sim_k$.

Theorem

Let $\mathcal{A}$, $\mathcal{B}$ be Kripke structures, with $a \in A$ and $b \in B$, and $k > 0$. The following are equivalent:

1. *There is a homomorphism $f : \mathbb{M}_k(\mathcal{A}, a) \to (\mathcal{B}, b)$.*
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It is not immediately obvious how to adapt the notion of p-morphism to match the finite levels of approximation $\sim_k$. The modal comonad offers an elegant solution.

Bisimulation approximants as spans

We say that $f : \mathbb{M}_k(\mathcal{A}, a) \to (\mathcal{B}, b)$ is a coKleisli *p*-morphism if $f^* : \mathbb{M}_k(\mathcal{A}, a) \to \mathbb{M}_k(\mathcal{B}, b)$ is a *p*-morphism of Kripke structures.

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For pointed Kripke structures $(\mathcal{A}, a)$ and $(\mathcal{B}, b)$: $a \sim_k b$ iff $(\mathcal{A}, a) \leftrightarrow_k^{\mathbb{M}} (\mathcal{B}, b)$.

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There is also the existential positive fragment, where we omit $\Box$ and negation.

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We have the modal fragment $\mathcal{M}_k$, which arises from the standard translation of (multi)modal logic into first-order logic, for formulas of modal depth $\leq k$.

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Theorem

1. *For all Kripke structures $\mathcal{A}$ and $\mathcal{B}$: $\mathcal{A} \equiv^{\exists \mathcal{M}_k} \mathcal{B} \iff \mathcal{A} \rightleftarrows^{\mathbb{M}}_k \mathcal{B}$.*
2. *For all Kripke structures $\mathcal{A}$ and $\mathcal{B}$: $\mathcal{A} \equiv^{\mathcal{M}_k} \mathcal{B} \iff \mathcal{A} \leftrightarrow^{\mathbb{M}}_k \mathcal{B}$.*
3. *For all image-finite Kripke structures $\mathcal{A}$ and $\mathcal{B}$: $\mathcal{A} \equiv^{\mathcal{M}_k(\#)} \mathcal{B} \iff \mathcal{A} \cong^{\mathbb{M}}_k \mathcal{B}$.*

Coalgebras and combinatorial parameters

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The lower the value of k, the less information available to Spoiler, and the easier it is for Duplicator to have a winning strategy. Equivalently, the easier it is to have a morphism from $\mathcal{A}$ to $\mathcal{B}$ in the co-Kleisli category.

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Restricting the access to $\mathcal{B}$ makes it *harder* for Duplicator to win the homomorphism game.

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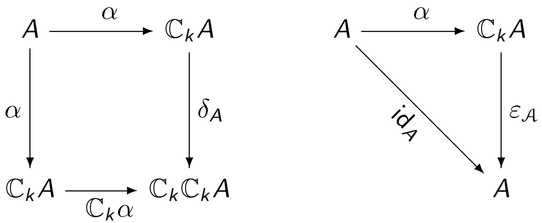

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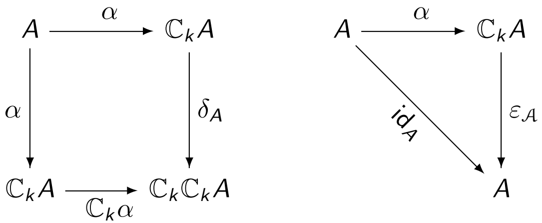

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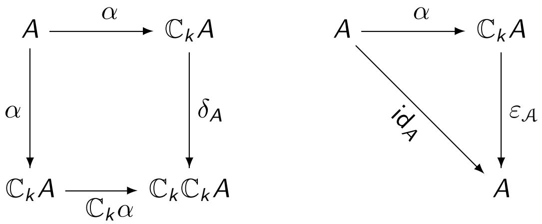

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Our use of indexed comonads $\mathbb{C}_k$ opens up a new kind of question for coalgebras. Given a structure $\mathcal{A}$, we can ask: what is the least value of k such that a $\mathbb{C}_k$-coalgebra exists on $\mathcal{A}$? We call this the *coalgebra number* of $\mathcal{A}$.

Coalgebra numbers

Theorem

- *For the pebbling comonad, the coalgebra number of $\mathcal{A}$ corresponds precisely to the tree-width of $\mathcal{A}$.*
- *For the Ehrenfeucht-Fraïssé comonad, the coalgebra number of $\mathcal{A}$ corresponds precisely to the tree-depth of $\mathcal{A}$.*
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We thus obtain categorical characterizations of these key combinatorial parameters.

Tree depth and the Ehrenfeucht-Fraïssé comonad

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Theorem

Let $\mathcal{A}$ be a finite σ-structure, and $k > 0$. There is a bijective correspondence between

1. $\mathbb{E}_k$*-coalgebras* $\alpha : \mathcal{A} \rightarrow \mathbb{E}_k\mathcal{A}$.
2. *Forest covers of* $\mathcal{G}(\mathcal{A})$ *of height* $< k$.

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A tree-decomposition of a graph $G = (V, \frown)$ is a tree $(T, \leq)$ together with a labelling function $\lambda : T \to \mathcal{P}(V)$ satisfying the following conditions:

- (TD1) for all $v \in V$, for some $x \in T$, $v \in \lambda(x)$;
- (TD2) if $v \frown v'$, then for some $x \in T$, $\{v, v'\} \subseteq \lambda(x)$;
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This parameter plays a fundamental role in combinatorics, algorithms and parameterized complexity.

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A k-pebble forest cover for a graph $G = (V, \frown)$ is a forest cover $(V, \leq)$ together with a pebbling function $p : V \to \mathbf{k}$ such that, if $v \frown v'$ with $v \leq v'$, then for all $w \in (v, v']$, $p(v) \neq p(w)$.

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Theorem

Let G be a finite graph. The following are equivalent:

1. *G has a tree decomposition of width $< k$.*
2. *G has a k-pebble forest cover.*

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Theorem

Let $\mathcal{A}$ be a finite σ-structure. There is a bijective correspondence between:

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For all finite structures $\mathcal{A}$: $\mathrm{tw}(\mathcal{A}) = \kappa^{\mathbb{P}}(\mathcal{A}) - 1$.

Indexed and graded structure

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Our comonads $\mathbb{E}_k$, $\mathbb{P}_k$, $\mathbb{M}_k$ are not merely discretely indexed by the resource parameter. In each case, there is a functor $(\mathbb{Z}^+, \leq) \to \mathsf{Comon}(\mathcal{R}(\sigma))$.

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The question is whether there are more interesting graded structures which arise naturally in considering richer logical and computational settings.

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Transfinite extensions are also possible. Similar constructions can be applied to the other comonads. This provides a basis for lifting the comonadic analysis to the level of infinite models.

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- Wider horizons: can we connect with significant meta-algorithms, such as decision procedures for guarded logics based on the tree model property, or algorithmic metatheorems such as Courcelle's theorem?

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- Need to understand better what makes these constructions work, and what the scope of these ideas are.
- Currently investigating the guarded fragment. Other natural candidates include existential second-order logic, and branching quantifiers and dependence logic.
- Wider horizons: can we connect with significant meta-algorithms, such as decision procedures for guarded logics based on the tree model property, or algorithmic metatheorems such as Courcelle's theorem?
- The wider issue: can we get Structure and Power to work with each other to address genuinely deep questions?

Envoi

Envoi

Let’s not forget to dream!