Causality, quantum channels, and Mealy machines

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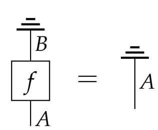
Abstract

We highlight a new and surprising connection between two very distinct notions of 'causality' that appear in the literature: the notion of causality coming from special relativity, which forbids the flow of information faster than the speed of light (a property also known as 'non-signalling'), and a coalgebraic notion of causality, which studies functions on streams (or generalisations thereof) whose output at a given time step can only depend on inputs from the past. Both of these notions deserve to be referred to as 'causality', as they explicitly require that information flows from past to future, and not vice-versa. We show that both the relativistic and coalgebraic notions of causality arise as instances of a single concept called 'one-way signalling', which can be defined in any symmetric monoidal category equipped with chosen 'discarding' maps.

In particularly well-behaved categories, it is furthermore the case that one-way signalling processes factorise in such a way that the flow of information from past to future becomes explicit. We show that this factorisation instantiates to two well-known constructions in totally different fields: the semi-localisation of a bipartite quantum channel and the construction of a Mealy machine from a causal function using stream derivatives.

In [3], Kissinger and Uijlen introduced the notion of a *precausal category*, which is intended to be a category with just enough structure to make meaningful, logical statements about the causal relationships between systems interacting in space and time. Precausal categories are symmetric monoidal categories with some additional structure, including chosen *discarding* morphisms $d_A : A \to I$ for each system A (where I is the monoidal unit). These enable one to single out the so-called *causal processes*, which are morphisms $f : A \to B$ satisfying $d_B \circ f = d_A$. As is common in monoidal categories, we switch to string diagram notatation [5], where this equation becomes:

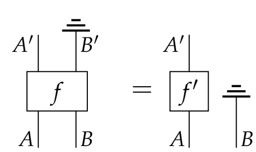


(1)

Note we draw inputs on the bottom and outputs on the top, and use the 'ground' symbol from electical circuits to represent discarding. This equation captures the intuition that, if we disregard the output of a process, then it doesn't matter which process occurred. Or, put another way, the *only* influence f has is on its output. This can be seen as f having no 'back action' or being 'side-effect free'.

We can also make finer-grained statements about the causal relationships between inputs and outputs of a process. For instance, consider a black box which takes an input A and produces an output A', then *later* takes an input B and produces an output B'. Then it should be case that the input B can only affect the output B'. Hence, if we discard B', the output of the overall black-box process $f : A \otimes B \to A' \otimes B'$ no longer depends on B. That is:

$\exists\, f'$ *causal.*

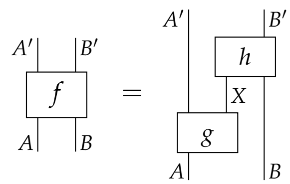


(2)

This is stronger than equation (1), since we no longer have to discard *all* of the outputs of f to discard some of the inputs. Since it only allows data (i.e. a 'signal') to pass from the (A, A') pair to the (B, B'), and not vice-versa, such processes are called *one-way signalling* [3]. If (2) and its symmetric equation—interchanging the roles of (A, A') and (B, B')—are satisfied, such a process is called *non-signalling*. The latter have been studied extensively in the foundations of physics, as they model precisly the situation of two distant observers, Alice (A, A') and Bob (B, B') who are so far apart, they cannot communication due to the 'speed limit' imposed by special relativity (i.e. no information can be transmitted faster than the speed of light).

Another way in which we can say the pair (A, A') is in the past of (B, B') is via a factorisation of f:

$\exists\, g, h$ *causal*.



(3)

That is, rather than making an external constraint on f, we can require that f has some internal structure. Namely, first a process g happens, which produces a the first output, and possibly some 'hidden' memory X, which feeds into h to produce the second output.

In any symmetric monoidal category with discarding, (3) $\Longrightarrow$ (2), simply by discarding the output of h and applying equation (1). However, the converse is the interesting direction. Given a process that satisfies (2), there is no way *a priori* to construct processes g and h which explicitly witness the flow of information from (A, A') to (B, B'). In [3], the authors showed that any precausal category satisifies (2) $\Longrightarrow$ (3).

As an example coming from physics, we can fix our category to be **CPM**, the category whose objects are complex vector spaces of the form $\mathcal{L}(H) := \hom_{\mathbf{Vect}}(H, H)$ and whose morphisms are completely positive maps $\Phi : \mathcal{L}(H) \to \mathcal{L}(K)$. Then morphisms satisfying (1) are the *trace-preserving* completely-positive maps, also known as *quantum channels*, which are the central object of study in quantum information theory. In [2], Eggeling et al showed that, for any quantum channel satisfying (2), one can construct a decomposition (3) using a technique which relies on the essential uniqueness of purification for quantum channels. They call a channel with decomposition (3) *semi-localisable*. The fact that any one-way signalling quantum channel is semi-localisable plays an important role e.g. in the study of quantum causal networks [1].

Now, if we fix our category to **Set**, we can consider maps on streams $f : A^\omega \to B^\omega$. For any n, we can employ the canonical isomorphisms $A^\omega \cong A^n \times A^\omega$ and $B^\omega \cong B^n \times B^\omega$ to consider f as a function with two inputs and two outputs, i.e. $f : A^n \times A^\omega \to B^n \times B^\omega$.

Since **Set** is cartesian, the only choice for discarding is the terminal map. Hence $\mathrm{id}_{A^n} \times d_{A^\omega}$ is just the projection onto the first n elements of a stream $s \mapsto s|_n$. From this, equation (2) to f yields:

$$\exists f' : A^n \to B^n . \forall s \in A^\omega . f(s)|_n = f'(s|_n)$$

Since this holds for all n, this exactly recovers the definition of a *causal function* on streams.

We could similarly consider functions f which factor as in (3) for all n, but we can say this more compactly with an equivalent, co-inductive definition. Say $f : A \times A^\omega \to B \times B^\omega$ is *ω-semi-localisable* if it factorises as in (3) and for all $x \in X$, $h(x, -) : A^\omega \to B^\omega$ is furthermore ω-semi-localisable.

Again, it is immediately clear that (3) $\Longrightarrow$ (2), but the converse (using the coinductive version of (3) given above) is saying something non-trivial about causal functions. Namely: it says that the set of all causal functions $C[A^\omega, B^\omega]$ is the final coalgebra for the Mealy machine functor $M(-) = (B \times -)^A$.

The coalgebra structure map $\phi : C[A^\omega, B^\omega] \to (B \times C[A^\omega, B^\omega])^A \cong B^A \times C[A^\omega, B^\omega]^A$ gives us all of the data we need to construct decomposition (3) from a causal function f. For $\phi(f) = (g', h')$, we let $X := A$, $g(a) := (g'(a), a)$ and $h(a, b) := h'(a)(b)$. Intuitively, this decomposes f into its action g' at a single time step and a family of causal functions $\{h'(a) | a \in A\}$ which continue acting like f starting at time step 2, given an a was input before. Furthermore, as long as A is a non-empty set, this map can be built explicitly using restriction, concatenation, and derivatives of streams. This is precisely the canonical construction of a Mealy machine from stream derivatives done e.g. in [4].

References

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