A coalgebraic take on regular and ω-regular behaviour for systems with internal moves

Tomasz Brengos

Faculty of Mathematics and Information Science
Warsaw University of Technology
t.brengos@mini.pw.edu.pl

One of the most fundamental state-based structures considered in computer science literature is a non-deterministic automaton. The class of finite languages accepted by this type of machine with a finite state-space is known under the name of *regular languages*. On the other hand, these systems have a natural infinite semantics which is given in terms of infinite input satisfying the so-called Büchi acceptance condition (or BAC in short). The condition takes into account the terminal states of the automaton and requires them to be visited infinitely often. It is a common practise to use the term *Büchi automaton* in order to refer to an automaton whenever its infinite semantics is taken into consideration. The class of infinite languages accepted by finite non-deterministic Büchi automata can be characterized by *Kleene theorem for ω-regular languages*. Roughly speaking, any such language can be represented in terms of regular languages and the infinite iteration operator $(-)^\omega$. Although, the standard type of input of a Büchi automaton is the set of infinite words over a given alphabet, other types (e.g. trees) are also commonly studied and suitable variants of the Kleene theorem hold. This begs the question of a unifying framework these systems can be put in and reasoned about on a more abstract level so that the analogues of Kleene theorems for (ω-)regular input are derived.

The aim of the talk is to present a framework as above for coalgebras with internal moves[1]. The approach from [2] suggests that systems with silent steps should be defined as coalgebras whose type is a monad. Although originally, systems with internal moves were modelled as coalgebras $X \to T(FX+X)$ for a monad T and an endofunctor F, such systems can be embedded into coalgebras $X \to TF^*X$, where F^* is the free monad over F and TF^* itself carries a monadic structure [2]. Unfortunately, the monad TF^* was only tailored to model finite behaviour and is insufficient to cover infinite behaviour. Hence, in the first part of the talk we focus on a description of a monad suitable for our purposes.

The construction of a suitable monad is based on an observation that if an endofunctor $F : \mathsf{C} \to \mathsf{C}$ *lifts* to the Kleisli category for a monad T then the monad T lifts to a monad $\widehat{T} : \mathsf{Alg}(F) \to \mathsf{Alg}(F)$ on the category of F-algebras. The same is true if we replace $\mathsf{Alg}(F)$ with the category $\mathsf{Alg}_B(F)$ of *Bloom algebras* [1]. The free objects in $\mathsf{Alg}_B(F)$ often exist and are combinations of the free F-algebras and the final F-coalgebra [1]. Hence, the monad TF^∞ that is suitable to model (in)finite behaviour of systems is defined by composing the

[1] A coalgebraic framework for Büchi automata has been recently developed [3], but it does not take invisible moves into the account and does not reason about (ω-)regular input

adjunctions $\mathsf{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{Alg}_B(F) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{K}l(\widehat{T}_B)$ [2]. For $T = \mathcal{P}$ and $F = \Sigma \times \mathcal{I}d$ we obtain $TF^\infty = \mathcal{P}(\Sigma^* \times \mathcal{I}d + \Sigma^\omega)$.

In the second part of the talk we present a categorical framework to reason about infinite behaviour with BAC. Any non-deterministic (Büchi) automaton without an initial state can be modelled as a pair $(\alpha : X \to \mathcal{P}(\Sigma \times X), \mathfrak{F} \subseteq X)$, where $\mathfrak{F}$ is the set of *terminal states*. We can extend the codomain of α and consider this map as $\alpha : X \to \mathcal{P}(\Sigma^* \times X + \Sigma^\omega)$ (i.e. an endomorphism in the Kleisli category for our monad). Similarly, $\mathfrak{F}$ is uniquely determined by $\mathfrak{f}_\mathfrak{F} : X \to \mathcal{P}(\Sigma^* \times X + \Sigma^\omega); x \mapsto$ **if** $x \in \mathfrak{F}$ **then** $\{(\varepsilon, x)\}$ **else** $\varnothing$. This means that any such automaton is determined by the pair $(\alpha, \mathfrak{f}_\mathfrak{F})$ of endomorphisms in the Kleisli category for the above monad. This category is ordered by a complete order, is left distributive and, hence, the following fixpoint operators are well defined: $\alpha^* = \mu x.id \vee x \cdot \alpha : X \to X$ and $\beta^\omega = \nu x.x \cdot \beta : X \to 0$ for $\beta : X \to X$. It can be proven that the map $||(\alpha, \mathfrak{F})|| : X \to \mathcal{P}(\Sigma^\omega)$ (i.e. $||(\alpha, \mathfrak{F})|| : X \to 0$ in the Kleisli category) which assigns to any state $x \in X$ the infinite language with BAC it accepts is given by $||(\alpha, \mathfrak{F})|| = (\mathfrak{f}_\mathfrak{F} \cdot \alpha^*)^\omega$. This suggests a general definition of the infinite behaviour with BAC for an arbitrary pair of endomorphisms $(\alpha, \mathfrak{f})$ in the Kleisli category for TF^∞, where $(-)^*$ and $(-)^\omega$ are well defined.

Now, for any natural numbers $m, n = 0, 1, \ldots$ put $[m] = \{1, \ldots, m\}$ and define $\mathfrak{Reg}(m, n)$ to be the set of morphisms of the form $j \cdot \mathfrak{f}_\mathfrak{F} \cdot \alpha^* \cdot i$, where $i : [m] \to [k]$ and $j : [k] \to [n]$ are Set-maps, $\mathfrak{F} \subseteq [k]$ and $\alpha : [k] \to T(F[k] + [k])$. Finally, put $\omega\mathfrak{Reg}$ to include maps $(\mathfrak{f}_\mathfrak{F} \cdot \alpha^*)^\omega \cdot i$ for a Set map $i : [1] \to [k]$ and $\alpha : [k] \to T(F[k] + [k])$ with $\mathfrak{F} \subseteq [k]$. It can be shown that for $T = \mathcal{P}$ and $F = \Sigma \times \mathcal{I}d$, the set $\mathfrak{Reg}(1, 1)$ and $\omega\mathfrak{Reg}$ are exactly the sets of regular and ω-regular languages respectively in the classical sense. Although it was sufficient for non-deterministic Büchi automata to express behaviours from $\omega\mathfrak{Reg}$ in terms of $\mathfrak{Reg}(1, 1)$ and $(-)^\omega$ it is not enough in general. Hence, we have to consider behaviours from $\mathfrak{Reg}(m, n)$ in order to state the coalgebraic Kleene theorem for ω-regular input. We discuss the conditions under which the following holds.

Theorem 1 (Kleene theorem for (ω-)regular behaviour). $\mathfrak{Reg}$ *forms an ordered Lawvere theory which is closed under finite joins,* $(-)^*$ *and it is the smallest subtheory of the Lawvere theory associated with the monad* TF^∞ *that contains all* $\alpha : [k] \to T(F[k] + [k])$ *and is closed under finite joins and* $(-)^*$*. Moreover,* $\omega\mathfrak{Reg} = \{[r_1, \ldots, r_n]^\omega \cdot r \mid r, r_i \in \mathfrak{Reg}(1, n)\}$.

References

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[2] Here, $\widehat{T}_B$ denotes a lifting of T to the category of $\mathsf{Alg}_B(F)$ Bloom F-algebras