A coalgebraic take on regular and ω-regular behaviour for systems with internal moves

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Our goal

Goal of the talk

Put infinite behaviour with Büchi acceptance condition into a coalgebraic framework...

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Wait...

So... we are doing the same as Urabe, Shimizu, Hasuo (CONCUR'16).

Yes, but in a different manner!

Our primary interest

Kleene th. for regular langauges

The set of regular languages for NA is closed under $\cup$, $\cdot$, $\varnothing$, $\{\varepsilon\}$ and $(-)^*$. Moreover, it is the smallest set of languages which contains $\{a\}$ and is closed under these operations.

Kleene th. for ω-regular languages

The ω-regular languages for Büchi automata ($=NA$) are of the form

$$\bigcup_{i=1}^{n} R_i^{\omega} \cdot L_i \text{ for regular languages } R_i, L_i$$

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Non-deterministic automata with ε-transitions

Definition

An ε-*NA* is a tuple $(X, \Sigma_\varepsilon, \to, \mathfrak{F})$, where X is a set of *states*, Σ is a finite set *alphabet letters* and $\to \subseteq X \times \Sigma_\varepsilon \times X$ and $\mathfrak{F} \subseteq X$ is the set of *terminal states*.

Any $\varepsilon - NA$ can be viewed as a *coalgebra*, i.e. a map:

$$\alpha' : X \to \mathcal{P}(\Sigma_\varepsilon \times X + 1);$$

where $1 = \{\checkmark\}$. However

Note

We can also view it as a pair $(\alpha, \mathfrak{F})$, where

$$\alpha : X \to \mathcal{P}(\Sigma_\varepsilon \times X).$$

From our previous work...

Any $\alpha : X \to \mathcal{P}(\Sigma_\varepsilon \times X)$ is a *labelled transition system with ε-moves*. Other known systems with internal moves:

- Segala systems,
- fully probabilistic systems,
- ...

$$\frac{\text{Systems with internal moves}}{\dfrac{\text{Coalgebras over a monad } X \to TX}{\text{Endomorphisms in the Kleisli category for T}}}$$

What to do with LTS?

Strategy 1

Introduce a monad structure on $\mathcal{P}(\Sigma_\varepsilon \times \mathcal{I}d)$

Strategy 2

Embed the functor $\mathcal{P}(\Sigma_\varepsilon \times \mathcal{I}d)$ into $\mathcal{P}(\Sigma^* \times \mathcal{I}d)$ which is a monad.

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Strategy 3

Embed the functor $\mathcal{P}(\Sigma_\varepsilon \times \mathcal{I}d)$ into $\mathcal{P}(\Sigma^* \times \mathcal{I}d + \Sigma^\omega)$ which is a monad: For $f : X \to \mathcal{P}(\Sigma^* \times Y + \Sigma^\omega)$ and $g : Y \to \mathcal{P}(\Sigma^* \times Z + \Sigma^\omega)$ the map $g \cdot f : X \to \mathcal{P}(\Sigma^* \times Z + \Sigma^\omega)$ is:

$$x \stackrel{\sigma}{\to}_{g \cdot f} z \iff \exists y \text{ s.t. } x \stackrel{\sigma_1}{\to}_f y \text{ and } y \stackrel{\sigma_2}{\to}_g z, \text{ where } \sigma = \sigma_1\sigma_2 \in \Sigma^*,$$

$$x \downarrow_{g \cdot f} v \iff x \downarrow_f v \text{ or } x \stackrel{\sigma}{\to}_f y,\ y \downarrow_g v' \text{ and } v = \sigma v' \in \Sigma^\omega.$$

Let's use the last strategy

$$\dfrac{\dfrac{\alpha : X \to \mathcal{P}(\Sigma_\varepsilon \times X)}{\alpha : X \to \mathcal{P}(\Sigma^* \times X + \Sigma^\omega)}}{\alpha : X \to X \text{ is an endo in } \mathcal{K}l(\mathcal{P}(\Sigma^* \times \mathcal{I}d + \Sigma^\omega))}$$

Interesting observation

Any subset $\mathfrak{F} \subseteq X$ may be encoded as an endomorphism in $\mathcal{K}l(\mathcal{P}(\Sigma^* \times \mathcal{I}d + \Sigma^\omega))$:

$$\mathfrak{f}_\mathfrak{F} : X \to \mathcal{P}(\Sigma_\varepsilon \times X); x \mapsto \begin{cases} \{(\varepsilon, x)\} & \text{if } x \in \mathfrak{F}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Non-deterministic automata as a pair of endomorphisms

Observation 2

Any non-deterministic automaton can be viewed as a pair of endomorphisms in the Kleisli of $\mathcal{P}(\Sigma^* \times \mathcal{I}d + \Sigma^\omega)$:

$$(\alpha, \mathfrak{f}_{\mathfrak{F}}).$$

How do we derive (in)finite behaviour of $(\alpha, \mathfrak{f}_{\mathfrak{F}})$?

Basic properties of the Kleisli category

The Kleisli of $\mathcal{P}(\Sigma^* \times \mathcal{I}d + \Sigma^\omega)$ is:

- order enriched by $f \leq g \iff f(x) \subseteq g(x)$ for any $x \in X$,
- the ordering is complete,
- it is left distributive, i.e. $f \cdot (g \vee h) = f \cdot g \vee f \cdot h$.

This allows us to consider for any endo $\alpha : X \to X$ the maps $\alpha^* : X \to X$ and $\alpha^\omega : X \to 0$:

$$\alpha^* = \mu x.(\mathrm{id} \vee x \cdot \alpha) \text{ and } \alpha^\omega = \nu x.x \cdot \alpha.$$

Finite behaviour

Finite behaviour of $(\alpha, \mathfrak{f}_{\mathfrak{F}})$

$$! \cdot \mathfrak{f}_{\mathfrak{F}} \cdot \alpha^* = X \stackrel{\alpha^*}{\to} X \stackrel{\mathfrak{f}_{\mathfrak{F}}}{\to} X \stackrel{!}{\to} 1 \text{ in Klesli}$$

Explanation

$$x \stackrel{\alpha^*}{\mapsto} \{(a_1 a_2 \ldots a_n, y) \mid x \stackrel{a_1}{\to} x_1 \ldots x_{n-1} \stackrel{a_n}{\to} x_n = y\} \cup \{(\varepsilon, x)\}$$

$$x \stackrel{\mathfrak{f} \cdot \alpha^*}{\mapsto} \{(a_1 a_2 \ldots a_n, y) \mid x \stackrel{a_1}{\to} x_1 \ldots x_{n-1} \stackrel{a_n}{\to} x_n = y \text{ and } y \in \mathfrak{F}\} \cup \{(\varepsilon, x) \mid x \in \mathfrak{F}\}.$$

$$x \stackrel{! \cdot \mathfrak{f} \cdot \alpha^*}{\mapsto} \{(w, 1) \mid w \text{ is accepted by the automaton } (\alpha, \mathfrak{F})\}.$$

Side note

The same finite behaviour would be obtained in the Kleisli of $\mathcal{P}(\Sigma^* \times X)$.

Why should we bother with $\mathcal{P}(\Sigma^* \times \mathcal{I}d + \Sigma^\omega)$?

Infinite behaviour with BAC

Let $(\alpha, \mathfrak{F})$ be an automaton without ε-transitions. Then

Infinite behaviour of $(\alpha, \mathfrak{F})$ with BAC

$$(\mathfrak{f}_{\mathfrak{F}} \cdot \alpha^{+})^{\omega} \text{ in Klesli}$$

In the above

$$\alpha^* = \mu x.(\mathrm{id} \vee x \cdot \alpha), \quad \alpha^{+} = \alpha^* \cdot \alpha \text{ and } \beta^{\omega} = \nu x.x \cdot \beta.$$

Explanation

For $\beta : X \to \mathcal{P}(\Sigma \times X)$ we have $\beta^{\omega} : X \to \mathcal{P}(\Sigma^{\omega})$:

$$\beta^{\omega}(x) = \{(a_1, a_2, \ldots) \mid x \xrightarrow{a_1} x_1 \xrightarrow{a_2} x_2 \ldots\}$$

Question 1.

We embedded $\mathcal{P}(\Sigma_\varepsilon \times \mathcal{I}d)$ into $\mathcal{P}(\Sigma^* \times \mathcal{I}d + \Sigma^\omega)$.

Problem

Can we do the same with TF_ε for a monad T and $F_\varepsilon = F + \mathcal{I}d$?

Goal

We want to embed TF_ε into TF^∞, where

$F^\infty =$ the combination of free F-algebra and final coalgebra

and TF^∞ carries a monadic structure.

Answer to Q1

Take $\mathsf{Alg}_B(F)$ the category of Bloom algebras of type F.

Fact (Adámek, Haddadi, Milius, et al. 2014)

The free Bloom F-algebra is the combination of of the free F-algebra and the final coalgebra.

We get a monad F^∞ as the consequence of: $\mathsf{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathsf{Alg}_B(F)$.
What about TF^∞?

Fact

If F lifts to $\mathcal{K}l(T)$ then the monad T lifts to a monad $\bar{T}_B$ on $\mathsf{Alg}_B(F)$.

$$\mathsf{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathsf{Alg}_B(F) \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathcal{K}l(\bar{T}_B).$$

Examples...

Ex 1

If $F = \Sigma \times \mathcal{I}d$ and $T = \mathcal{P}$ then the monad $TF^{\infty} = \mathcal{P}(\Sigma^* \times \mathcal{I}d + \Sigma^{\omega})$

Ex 2

If $F = \Sigma \times \mathcal{I}d^2$ then $F^{\infty}X = T_{\Sigma}X$ is the monad of complete binary finite and infinite trees with nodes in Σ and finitely many leaves, all in X. Then $\mathcal{P}T_{\Sigma}(-)$ is the monad of subsets of such trees.

In general, for a pair ($T =$ monad, $F =$ functor) on Set we consider the monad TF^{∞} and define finite and infinite behaviour of $(\alpha, \mathfrak{F})$ according to:

$$! \cdot \mathfrak{f}_{\mathfrak{F}} \cdot \alpha^* \text{ and } (\mathfrak{f}_{\mathfrak{F}} \cdot \alpha^+)^{\omega}.$$

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Question 2.

$(-)^*$ and $(-)^\omega$ are operators on endomorphisms in Kleisli. How are they related to the classical operators on languages for finite automata?

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What about tree automata?

Finite tree automata are: $(\alpha : [n] \to \mathcal{P}(\Sigma \times [n] \times [n]), \mathfrak{F} \subseteq [n])$, where $[n] = \{1, 2, \ldots, n\}$ is the state-space. Let (ω) $\mathfrak{Reg}$ = the set of (in)finite tree languages accepted by finite tree automata. We have to introduce $\mathfrak{Rat}_n$.

Definition

$\mathfrak{Rat}_n$ is defined to be the smallest set of tree languages with variables in $[n]$ such that it contains $\{\varepsilon\}$, $\{i\}$ for $i \in [n]$, is closed under $\cup$, and $T_1, \ldots T_n \in \mathfrak{Rat}_m$, $T \in \mathfrak{Rat}_n$ we have:

$$[T_1, \ldots, T_n] \cdot T \in \mathfrak{Rat}_m \text{ and } T^{*,i} \in \mathfrak{Rat}_n.$$

Classical Kleene th. for tree languages

$$\mathfrak{Reg} = \mathfrak{Rat}_1$$
$$\omega\mathfrak{Reg} = \{[T_1, \ldots T_n]^\omega \cdot T \mid T, T_i \in \mathfrak{Rat}_n\}.$$

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The general picture

1. Start with a finite (T, F)-automaton $(\alpha : X \to TFX, \mathfrak{F} \subseteq X)$,
2. Consider the monad TF^∞ and look at $(\alpha, \mathfrak{F})$ as a pair of endomorphisms $(\alpha, \mathfrak{f})$ in the Kleisli for TF^∞,
3. define finite and infinite behaviour with BAC:
$$! \cdot \mathfrak{f} \cdot \alpha^* \qquad (\mathfrak{f} \cdot \alpha^+)^\omega.$$
4. define $\mathfrak{Rat}_n$ to be the smallest set of elements from $TF^\omega[n]$ containing 0, i and being closed under $\vee$ and
$$[r_1, \dots r_n] \cdot r \in \mathfrak{Rat}_m \text{ and } r^{*_i} \in \mathfrak{Reg}_n \text{ if } r \in \mathfrak{Rat}_n, r_i \in \mathfrak{Rat}_m.$$

Kleene theorem

$$\mathfrak{Reg} = \mathfrak{Rat}_1$$
$$\omega\mathfrak{Reg} = \{[r_1, \dots r_n]^\omega \cdot r \mid r, r_i \in \mathfrak{Rat}_n\}.$$

Thank you!