On Retracts of Algebras with Iteration

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Abstract. We show that iteration-congruent retracts of (completely) iterative algebras are (complete) Elgot algebras. Conversely, for an iteratable endofunctor H, every (complete) Elgot H-algebra arises as an iteration-congruent retract of a (completely) iterative H-algebra.

In a recent work, Goncharov *et al.* [5] study a relationship between different kinds of monads with iteration. In particular, they show that iteration-congruent retracts of completely iterative monads [1] yield complete Elgot monads [3]. Conversely, provided certain final coalgebras exist, every complete Elgot monad arises this way, that is, as an iteration-congruent retract of a completely iterative monad. In this note, we present similar results for algebras with iteration: (complete) Elgot algebras [2] and (completely) iterative algebras [1,6].

Let H be an endofunctor on a category $\mathscr{C}$ with binary coproducts. An *H-algebra with iteration* is a triple $\langle A, a : HA \to A, (\text{-})^\dagger \rangle$, where the $(\text{-})^\dagger$ operator assigns to every morphism $e : X \to A + HX$ a *solution*, that is, a morphism $e^\dagger : X \to A$ such that $e^\dagger = [\mathsf{id}, a] \cdot (\mathsf{id} + He^\dagger) \cdot e$. A *complete Elgot algebra* is an algebra with iteration in which the $(\text{-})^\dagger$ operator satisfies two additional axioms: *functoriality* and *compositionality* (see [2]). In what follows, given morphisms $e : X \to A + HX$ and $f : A \to B$, we write $f \bullet e$ for the morphism $(f + \mathsf{id}) \cdot e : X \to B + HX$.

Definition 1. *Let $\langle A, a, (\text{-})^\dagger \rangle$ be an H-algebra with iteration, and $\langle B, b \rangle$ be an H-algebra. We call a morphism $\rho : A \to B$ an* iteration-congruent retraction *if the following hold:*

1. *ρ is an algebra homomorphism $\langle A, a \rangle \to \langle B, b \rangle$,*
2. *ρ as a morphism in $\mathscr{C}$ has a section $\sigma : B \to A$,*
3. *ρ is iteration-congruent, that is, for all $e, f : X \to A + HX$, it is the case that $\rho \bullet e = \rho \bullet f$ implies $\rho \cdot e^\dagger = \rho \cdot f^\dagger$.*

Theorem 2. *Let $\langle A, a : HA \to A, (\text{-})^\dagger \rangle$ be a complete Elgot H-algebra, and $\langle B, b \rangle$ be an H-algebra. Then, given an iteration-congruent retraction $\rho : A \to B$, the algebra $\langle B, b \rangle$ can be given a complete Elgot structure with the solution of a morphism $e : X \to B + HX$ given as $e^\ddagger = \rho \cdot (\sigma \bullet e)^\dagger$. Moreover, in such a case ρ preserves solutions, that is, $(\rho \bullet e)^\ddagger = \rho \cdot e^\dagger$.*

A *completely iterative algebra* is an algebra $\langle A, a : HA \to A \rangle$ such that for a morphism $e : X \to A + HX$, there exists a unique solution $e^\dagger : X \to A$. Every completely iterative algebra, understood as an algebra with iteration, is a complete Elgot algebra. Thus, we obtain the following corollary of Theorem 2:

Corollary 3. *An iteration-congruent retract of a completely iterative algebra is a complete Elgot algebra.*

We also show that the converse holds if we assume an additional property of the endofunctor H. We say that H is *iteratable* [1] if the endofunctor $A+H(\text{-})$ has a final coalgebra for every object A. We write $H^\infty A$ to denote the carrier of such a final coalgebra. Importantly, if H is iteratable, each object A generates a free complete Elgot algebra $\mathbf{F}A = \langle H^\infty A, \tau, (\text{-})^\dagger \rangle$, which happens to be completely iterative (see [2] for a detailed description of these results).

Theorem 4. *If H is iteratable, then every complete Elgot H-algebra $\langle A, a, (\text{-})^\ddagger \rangle$ arises as an iteration-congruent retract of a completely iterative algebra. The retraction is given by the unique morphism from $\mathbf{F}A$, given as $\mathsf{out}^\ddagger : H^\infty A \to A$, where $\mathsf{out} : H^\infty A \to A + HH^\infty A$ is the action of the final coalgebra.*

An instance of such an iteration-congruent retraction can be found in Example 3.10 in [2]. Consider a complete lattice with a carrier A. Given a possibly infinite binary tree with labels from A in the leaves (that is, the carrier of the free completely iterative algebra of the endofunctor $X \mapsto X \times X$ on **Set** generated by A), the iteration-congruent retraction takes the join of all the leaves in the tree. This gives us a complete Elgot structure on the complete lattice A.

Adámek *et al.* [1,2] consider also non-complete versions of Elgot algebras and iterative algebras. For those, we assume that $\mathscr{C}$ is locally finitely presentable, and we require algebras with iteration to have solutions for morphisms $e : X \to A+HX$ if X is finitely presentable. The results shown in this note trivially hold in the non-complete version as well, since they do not rely on completeness and the construction of solutions does not require solving morphisms with different X's.

Theorem 4 is related to previous work [4] as follows. By [4, Theorem 5.7], the category of complete Elgot algebras is isomorphic to the category of (Eilenberg-Moore) H^∞-algebras, and so the retraction $H^\infty A \to A$ in question can be alternatively obtained as the H^∞-algebra structure on A. Conditions of Definition 1 are easily seen to be satisfied, e.g. (3) is due to the fact that any H^∞-algebra structure is always an H^∞-algebra morphism, and those isomorphically correspond to complete Elgot algebra morphisms.

References

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