On Algebras with Effectful Iteration

Stefan Milius, Jiří Adámek, and Henning Urbat

FRIEDRICH-ALEXANDER
UNIVERSITÄT
ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

Motivation

Functors $F : \mathscr{A} \to \mathscr{A}$ model behaviour types of state-based systems

Final F-coalgebra $\nu F =$ fully abstract behaviour domain for all F-coalgebras

Lambek's Lemma. $t : \nu F \to F(\nu F)$ is a fixed point of F.

What if we are only interested in „finite" coalgebras?

Then final semantics may not be satisfactory:

☹ Junk: νF contains behaviour not realized by any finite F-coalgebra.

☹ Confusion: two states may be identified even though they are not behaviourally equivalent on the level of finite coalgebras

Solution: replace νF by a coalgebra capturing precisely finite behaviours.

Motivation

FRIEDRICH-ALEXANDER
UNIVERSITÄT
ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

This talk: three fixed points for „finite-state“ behaviour

$$\varphi F \twoheadrightarrow \varrho F \twoheadrightarrow \vartheta F \rightarrowtail \nu F$$

locally ffg fixed point

rational fixed point

locally finite fixed point

Theorem. F is a proper functor on algebraic category iff Milius, CALCO 2017.

$$\varphi F \cong \varrho F \cong \vartheta F \rightarrowtail \nu F$$

New Result. A universal property of φF as an F-algebra.

Examples of Rational Fixed Points

FX	coalgebras	νF	$\varrho F \cong \vartheta F$
$\{0,1\} \times X^\Sigma$	deterministic automata	$\mathcal{L} = 2^{\Sigma^*}$	regular languages
$\{0,1\} \times \mathcal{P}_\mathsf{f}(X)^\Sigma$	non-determ. automata	branching behavior (up to bisimilarity)	finite state branching behaviors
$\coprod_{n \in \mathbb{N}} \Sigma_n \times X^n$	Σ-automata	Σ-trees	rational Σ-trees
$k \times X$	stream automata	k^ω	eventually periodic streams

$\mathscr{A} = \mathsf{Vec}_k$

$FX = k \times X$		k^ω	rational streams

$\mathscr{A} = \mathbb{S}$-Mod for $\mathbb{S}$ a Noetherian semiring

$FX = \mathbb{S} \times X^\Sigma$	weighted automata	formal power-series $\mathbb{S}^{\Sigma^*}$	rational formal power series

carrier $\mathbb{S}^n$

Further examples …

Milius, Pattinson, Wißmann, FoSSaCS 2016.

$\ldots$ of ϑF (which are likely $\not\cong \varrho F$)

Rational formal power-series for every semiring

Context-free Languages

Real-time deterministic context-free languages
(= stack machine behaviours)

Context-free formal power series

The monad of Courcelle‘s algebraic trees

The Setting: Algebraic Categories

Assumptions.

- $\mathscr{A}$ an algebraic category, i.e. $\mathscr{A} = \mathsf{Set}^T$ for a finitary monad T on Set
- $F : \mathscr{A} \to \mathscr{A}$ finitary ~~and preserves reflexive coequalizers~~

for F a lifting:

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, "U"'] & \mathscr{A} \arrow[d, "U"] \\ \mathsf{Set} \arrow[r, "F_0"'] & \mathsf{Set} \end{tikzcd}$$

Examples. Precisely the finitary varieties of algebras, e.g. sets, monoids, groups, vector spaces, etc.

Three notions of „finite“ objects:

1. free finitely generated (ffg) algebras, i.e. TX for X a finite set
2. finitely presentable (fp) algebras, i.e. coequalizers $TX \rightrightarrows TY \twoheadrightarrow C$ for X, Y finite sets
3. finitely generated (fg) algebras, i.e. quotients $TX \twoheadrightarrow A$ for X a finite set

Remark: ffg $\Longrightarrow$ fp $\Longrightarrow$ fg but not conversely

Three ‚finite state' behaviour domains

$$\mathsf{Coalg}_{\mathsf{ffg}}\, F \hookrightarrow \mathsf{Coalg}_{\mathsf{fp}}\, F \hookrightarrow \mathsf{Coalg}_{\mathsf{fg}}\, F \hookrightarrow \mathsf{Coalg}\, F$$

all $TX \to FTX$ with X finite

all $A \to FA$ with A fp

all $A \to FA$ with A fg

Constructions.

$$\varphi F \xrightarrow{\zeta} F(\varphi F) := \mathsf{colim}(\mathsf{Coalg}_{\mathsf{ffg}}\, F \hookrightarrow \mathsf{Coalg}\, F)$$

$$\varrho F \xrightarrow{r} F(\varrho F) := \mathsf{colim}(\mathsf{Coalg}_{\mathsf{fp}}\, F \hookrightarrow \mathsf{Coalg}\, F)$$

$$\vartheta F \xrightarrow{\ell} F(\vartheta F) := \mathsf{colim}(\mathsf{Coalg}_{\mathsf{fg}}\, F \hookrightarrow \mathsf{Coalg}\, F)$$

Theorems.

1. All three coalgebras are fixed points of F.
2. The rational fixed point ϱF is the final locally fp coalgebra and the initial iterative algebra.
3. The locally finite fixed point ϑF is the final locally fg coalgebra and the initial fg-iterative algebra.

Urbat, CALCO 2017.

Adámek, Milius, Velebil, MSCS 2006.

Pattinson, Milius, Wißmann, FoSSaCS 2016.

Relationship of the Three

Milius, CALCO 2017.

Theorem. For F preserving injections and surjections:

1. $\varphi F \twoheadrightarrow \varrho F \twoheadrightarrow \vartheta F \rightarrowtail \nu F$
2. $\varphi F \cong \varrho F \cong \vartheta F \rightarrowtail \nu F$ if and only if F is proper.

Example. . . . where all four fixed points φF, ϱF, ϑF and νF differ.

An interesting locally ffg fixed point

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

Example. $\mathscr{A} =$ sets with one unary operation u

$F = \mathsf{Id} : \mathscr{A} \to \mathscr{A}$ identity functor

$\varrho F = \vartheta F = \nu F = 1$ are trivial

$\varphi F =$ colimit of all F-coalgebras $TX_0 \xrightarrow{\gamma} TX_0$ with X_0 finite

Free algebra on set of generators X_0:

$$TX_0 = \mathbb{N} \times X_0 \quad \text{with} \quad u(n,x) = (n+1,x)$$

$$TX_0 \xrightarrow{\gamma} TX_0 \quad \text{in } \mathscr{A} \quad \rightsquigarrow \quad X_0 \to \mathbb{N} \times X_0 \quad \text{in Set}$$

An interesting locally ffg fixed point

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

Two eventually periodic streams of natural numbers:

$$
\begin{array}{lll}
s = s_0, \ldots, s_k, & \overline{s}_1, \ldots, \overline{s}_p, & \overline{s}_1, \ldots, \overline{s}_p, \\
t = t_0, \ldots, t_l, & \overline{t}_1, \ldots, \overline{t}_p, & \overline{t}_1, \ldots, \overline{t}_p \quad \ldots
\end{array}
$$

are called equivalent if $\sum_{i<p} \overline{s}_i = \sum_{i<p} \overline{t}_i$.

For example $s = 5, 4, \quad 1, 2, 3, \quad 1, 2, \ldots$
$t = 1, 37, 42, \quad 2, 2, 2, \quad \ldots$

Carrier: $|\varphi F| =$ eq… classes of eventially periodic streams

Unary opn: …$, s_2, \ldots]) = [s_0 + 1, s_1 + 1, s_2 + 1, \ldots]$

Coalge… …ure: id : $\varphi F \to \varphi F$

…ality: every constant map is an F-coalgebra morphism from $TX \xrightarrow{\gamma} TX$ to $\varphi F \xrightarrow{\text{id}} \varphi F$.

What universal property does φF have?

Recap: Iterative Algebras

Definition.

An F-algebra $FA \xrightarrow{\alpha} A$ is called ~~iterative~~ fg-iterative if every equation morphism $X \xrightarrow{e} FX + A$ with X ~~fp~~ fg has a unique solution:

$$\begin{tikzcd}
X \arrow[r, dashed, "\exists! e^{\dagger}"] \arrow[d, "e"'] & A \\
FX + A \arrow[r] & FA + A \arrow[u]
\end{tikzcd}$$

However uniqueness fails for φF!

Theorem.

1. The … xed point, viewed as F-algebra $F(\varrho F) \to \varrho F$ … itial iterative algebra.
2. The locally finite fixed point, viewed as F-algebra $F(\vartheta F) \to \vartheta F$ is the initial fg-iterative algebra.

FFG-Elgot Algebras

Idea. In lieu of unique solutions assume a choice of canonical solutions.

Definition. An ffg-Elgot algebra is a triple $(A, \alpha, \dagger)$ where

- $FA \xrightarrow{\alpha} A$ is an F-algebra
- $\dagger$ assigns to every equation morphism $X \xrightarrow{e} FX + A$ with X ffg a solution $X \xrightarrow{e^\dagger} A$:

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ FX + A \arrow[r, "Fe^\dagger + A"'] & FA + A \arrow[u, "{[\alpha, A]}"'] \end{tikzcd}$$

subject to two natural axioms:

weak functoriality and compositionality:

Two natural axioms of dagger

Notation. 1. Given $X \xrightarrow{e} FX + Y$ and $Y \xrightarrow{h} A$ form

$$h \bullet e = (X \xrightarrow{e} FX + Y \xrightarrow{FX+h} FX + A)$$

Weak Functoriality.

Given equation morphisms $X \xrightarrow{e} FX + Z$, $Y \xrightarrow{f} FY + Z$ with X, Y, Z ffg:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "m"'] & FX + Z \arrow[d, "Fm+Z"] \\ Y \arrow[r, "f"'] & FY + Z \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} X \arrow[dr, "(h\bullet e)^\dagger"] \arrow[dd, "m"'] & \\ & A \\ Y \arrow[ur, "(h\bullet f)^\dagger"'] & \end{tikzcd}$$ for all $Z \xrightarrow{h} A$

Two natural axioms of dagger

Notation. 1. Given $X \xrightarrow{e} FX + Y$ and $Y \xrightarrow{h} A$ form

$$h \bullet e = (X \xrightarrow{e} FX + Y \xrightarrow{FX+h} FX + A)$$

2. Given $X \xrightarrow{e} FX + Y$ and $Y \xrightarrow{f} FY + A$ form

$$f \blacksquare e = (X + Y \xrightarrow{[e,\mathsf{inr}]} FX + Y \xrightarrow{FX+f} FX + FY + A \xrightarrow{\mathsf{can}+A} F(X+Y) + A)$$

Compositionality.

For every $X \xrightarrow{e} FX + Y$ and $Y \xrightarrow{f} FY + A$ with X, Y ffg:

$$(f \blacksquare e)^\dagger = [(f^\dagger \bullet e)^\dagger, f^\dagger] : X + Y \to A \qquad \text{where } f^\dagger : Y \to A$$

Why „Effectful Iteration"?

Suppose that $F : \mathsf{Set}^T \to \mathsf{Set}^T$ is a lifted set functor:

$$\begin{tikzcd} \mathsf{Set}^T \arrow[r, "F"] \arrow[d] & \mathsf{Set}^T \arrow[d] \\ \mathsf{Set} \arrow[r, "F_0"] & \mathsf{Set} \end{tikzcd}$$

Equivalently we have a distributive law $TF_0 \xrightarrow{\lambda} F_0 T$
of the monad T over the functor F_0

An F-algebra is given by two maps $F_0 A \xrightarrow{a} A \xleftarrow{\alpha} TA$
where α is an Eilenberg-Moore algebra and $\alpha \cdot Ta = a \cdot F_0\alpha \cdot \lambda_A$.

An effectful recursive equation is a map $X_0 \to T(F_0 X_0 + A)$.

finite set

disjoint union

This gives an equation morphism $TX_0 \to FTX_0 \oplus A$ in Set^T

coproduct in Set^T

whose solution $TX_0 \to A$ corresponds to a map $X_0 \xrightarrow{e_0^\dagger} A$ with

$$\begin{tikzcd} X_0 \arrow[rr, "e_0^\dagger"] \arrow[dd, "e_0"'] & & A \\ & & TA \arrow[u, "\alpha"'] \\ T(F_0 X_0 + A) \arrow[rr, "T(F_0 e_0^\dagger + A)"] & & T(F_0 A + A) \arrow[u, "{T[a, A]}"'] \end{tikzcd}$$

Main Results

Theorem. The algebra $F(\varphi F) \to \varphi F$ is an ffg-Elgot algebra.

Definition. The category ffg-Elgot(F) has

objects $=$ ffg-Elgot algebras $(A, \alpha, \dagger)$ for F
morphisms $=$ solution-preserving morphisms $(A, \alpha, \dagger) \xrightarrow{h} (B, \beta, \ddagger)$:

$$
\begin{tikzcd}
 & A \arrow[d, "h"] \\
X \arrow[ur, "e^\dagger"] \arrow[dr, "(h\bullet e)^\ddagger"'] & \\
 & B
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
FA \arrow[r, "\alpha"] \arrow[d, "Fh"'] & A \arrow[d, "h"] \\
FB \arrow[r, "\beta"'] & B
\end{tikzcd}
$$

Theorems.

1. The algebra $F(\varphi F) \to \varphi F$ is the initial ffg-Elgot algebra for F.
2. The forgetful functor ffg-Elgot$(F) \to \mathscr{A}$, $(A, \alpha, \dagger) \mapsto A$ is monadic (in particular, it has a left-adjoint).

free ffg-Elgot algebra exist

Main Results

Construction of free ffg-Elgot algebras:

Given a free object Y of $\mathscr{A}$ form the locally ffg fixed point of $F(-)+Y$:

$$\Phi Y := \varphi(F(-)+Y) \quad \text{with} \quad \Phi Y \underset{[\alpha_Y,\eta_Y]}{\rightleftarrows} F(\Phi Y)+Y \quad \text{and} \quad \dagger$$

Define a solution operator w.r.t. F: given $X \xrightarrow{e} FX + A$ let

$$e^{\ddagger} := (X \xrightarrow{e} FX + A \xrightarrow{[\mathsf{inl},\mathsf{inr}]} FX + Y + A)^{\dagger}$$

Theorem. Then $(\Phi Y, \alpha_Y, \ddagger)$ is a free ffg-Elgot algebra on Y with the universal morphism $Y \xrightarrow{\eta_Y} \Phi Y$.

Remark. For arbitrary objects Y in $\mathscr{A}$ this may not hold.

Conclusions

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

- In algebraic categories three fixed points capture the behaviour of „finite“ coalgebras:

$$\varphi F \twoheadrightarrow \varrho F \twoheadrightarrow \vartheta F \rightarrowtail \nu F$$

locally ffg fixed point | rational fixed point | locally finite fixed point

- In general they are different, but for proper functors they coincide and are fully abstract for standard behavioural equivalence.

- The locally ffg fixed point φF is characterized as the initial ffg-Elgot algebra for F, but fails to have a universal property as an F-coalgebra.

- Free ffg-algebras exist and are obtained as expected for free objects

- ffg-Elgot algebras are monadic over the base category.

Future Work

- Find a coalgebraic construction for arbitrary free ffg-Elgot algebras.
- Investigate the properties of the free ffg-Elgot algebra monad.
- Show that the composite of the following functors is monadic:

$$\mathsf{ffg\text{-}Elgot}(F) \hookrightarrow \mathsf{Alg}(F) \to \mathscr{A} \to \mathsf{Set}$$