Steps and Traces

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Trace semantics

I mean, for instance:

- language of a non-deterministic automaton
- (finite) traces of a labelled transition systems
- language accepted by a tree automaton
- ...

Coalgebraic bisimilarity is stronger than the intended (finite) trace semantics

This makes sense: captures behavioural equivalence – but traces abstract away from certain things

Coalgebraic trace semantics

Many different approaches . . .

- "Branching type" a monad T (powerset for non-determinism in LTSs and automata, distribution for probabilities, etc)
  - move to coalgebras in $\mathcal{K}\ell(T)$, use finality (Hasuo et al.)
  - move to coalgebras in $\mathcal{EM}(T)$, use finality ('determinisation', Silva et al.)
  - iteration-based approaches
- "Logic/tests" based:
  - move to *algebras*, use *initiality* (Klin/Rot, Pavlovic et al.)

- Finality (or initiality) in a different category in most approaches
- Some approaches more 'canonical', others more parametric

Corecursiveness and steps

Our work: two unifying ideas:

1. All approaches give trace semantics via a corecursive algebra on the set of 'languages'
2. In all cases, this corecursive algebra is constructed from a single abstract setting (adjunction together with a so-called step); using a final coalgebra in another category.

Corecursive algebra

An algebra $a\colon H(A) \to A$ is *corecursive* when for every coalgebra $c\colon X \to H(X)$ there is a coalgebra-to-algebra morphism:

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & A \\ {\scriptstyle c}\downarrow & & \uparrow{\scriptstyle a} \\ H(X) & \underset{H(f)}{\dashrightarrow} & H(A) \end{array}$$

Non-deterministic automata

$$
\begin{array}{ccc}
X & \overset{L}{\dashrightarrow} & 2^{A^*} \\
{\scriptstyle \langle o,d \rangle}\big\downarrow & & \big\uparrow{\scriptstyle a} \\
2 \times \mathcal{P}(X)^A & \xrightarrow[\mathrm{id}\times(\mathcal{P}L)^A]{} & 2 \times \mathcal{P}(2^{A^*})^A
\end{array}
$$

$$L(x)(\varepsilon) = o(x) \quad L(x)(aw) = \bigvee_{y \in d(x)(a)} L(y)(w)$$

The algebra a encodes the disjunction/union $\bigvee$

Alternating automata

$$
\begin{array}{ccc}
X & \overset{L}{- - - - - - - - \rightarrow} & 2^{A^*} \\
{\scriptstyle \langle o,d \rangle}\big\downarrow & & \big\uparrow{\scriptstyle a} \\
2 \times (\mathcal{P}\mathcal{P}X)^A & \xrightarrow[\mathrm{id} \times (\mathcal{P}\mathcal{P}L)^A]{} & 2 \times (\mathcal{P}\mathcal{P}(2^{A^*}))^A
\end{array}
$$

$$
L(x)(\varepsilon) = o(x) \quad L(x)(aw) = \bigvee_{S \in d(x)(a)} \bigwedge_{y \in S} L(y)(w)
$$

The algebra a encodes $\bigvee \bigwedge$

Tree automata

For $\Sigma \colon \mathbb{N} \to \mathbf{Set}$ a signature, define

$$H_\Sigma(X) = \coprod_{n \in \mathbb{N}} \Sigma(n) \times X^n$$

Tree automata:

$$\begin{array}{ccc}
X & \overset{L}{\dashrightarrow} & 2^{\Sigma^*} \\
{\scriptstyle f}\downarrow & & \uparrow{\scriptstyle a} \\
\mathcal{P}(H_\Sigma X) & \xrightarrow[\mathcal{P}H_\Sigma(L)]{} & \mathcal{P}(H_\Sigma(2^{\Sigma^*}))
\end{array}$$

- Σ^* the set of trees
- a again, a corecursive algebra

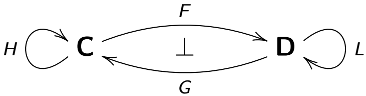
Trace semantics via corecursive algebras

Corecursive algebras arise from a final coalgebra in a different category, in:

- the 'Eilenberg-Moore' approach
- the 'logical' approach
- the 'Kleisli' approach

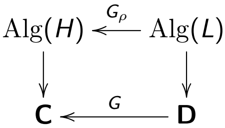
Unified by abstract setting of an adjunction and a "step"

General setting: step-and-adjunction



with a step: $HG \stackrel{\rho}{\Longrightarrow} GL$

Gives rise to a lifting:



$$G_\rho\Big(L(A) \xrightarrow{a} A\Big) \;:=\; \Big(HG(A) \xrightarrow{\rho} GL(A) \xrightarrow{G(a)} G(A)\Big).$$

Capretta/Uustalu/Vene: G_ρ preserves corecursive algebras


Step-induced algebra liftings
of right adjoints
preserve corecursiveness

General setting: step-and-adjunction

$$H \circlearrowright \mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D} \circlearrowleft L \qquad (F \dashv G)$$

with a step: $HG \overset{\rho}{\Longrightarrow} GL$

Theorem
Suppose that L has a final coalgebra $\zeta \colon \Psi \overset{\cong}{\Rightarrow} L(\Psi)$. Then for every H-coalgebra (X, c) there is a unique coalgebra-to-algebra map $c^\dagger$ in:

$$\begin{array}{ccc} X & \overset{c^\dagger}{\dashrightarrow} & G(\Psi) \\ {\scriptstyle c}\downarrow & & \uparrow{\scriptstyle G_\rho(\Psi, \zeta^{-1})} \\ H(X) & \overset{H(c^\dagger)}{\dashrightarrow} & HG(\Psi) \end{array}$$

We focus on instances where $c^\dagger$ captures traces, and call it the *trace semantics* map.

Eilenberg-Moore and Steps

1. A functor $B \colon \mathbf{C} \to \mathbf{C}$ with a final coalgebra $\zeta \colon \Theta \overset{\cong}{\Rightarrow} B(\Theta)$
2. A monad (T, η^T, μ^T) on $\mathbf{C}$
3. A lifting $\overline{B}$ of B to $\mathcal{EM}(T)$

$$BT \circlearrowright \mathbf{C} \underset{U}{\overset{\mathcal{F}}{\underset{\bot}{\rightleftarrows}}} \mathcal{EM}(T) \circlearrowleft \overline{B}$$

with

$$\rho \colon BTU \Longrightarrow U\overline{B} \quad \text{where}$$
$$\rho_{(X,a)} = \left(BTX \xrightarrow{Ba} BX\right)$$

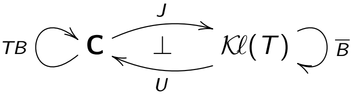
Gives rise to a corecursive algebra ℓ_{em}:

$$\begin{array}{ccc} X & \overset{\mathsf{em}_c}{\dashrightarrow} & \Theta \\ {\scriptstyle c}\downarrow & & \uparrow{\scriptstyle \ell_{\mathrm{em}}} \\ BT(X) & \underset{BT(\mathsf{em}_c)}{\dashrightarrow} & BT(\Theta) \end{array}$$

Transpose of em_c arises by 'determinisation'

Kleisli and Steps

1. A functor $B\colon \mathbf{C} \to \mathbf{C}$ with an initial algebra $\beta\colon B(\Psi) \xrightarrow{\cong} \Psi$.
2. A monad (T, η^T, μ^T) on $\mathbf{C}$
3. An extension $\overline{B}$ of B to $\mathcal{K}\ell(T)$
4. $(\Psi, J(\beta^{-1}))$ is a final $\overline{B}$-coalgebra.



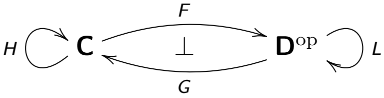
with

$$\rho\colon TBU \Longrightarrow U\overline{B} \quad \text{where } \rho_X = \left(TBTX \xrightarrow{T(\lambda)} T^2BX \xrightarrow{\mu^T} TBX\right)$$

$$\begin{array}{ccc} X & \overset{\mathrm{kl}_c}{\dashrightarrow} & T(\Psi) \\ {\scriptstyle c}\downarrow & & \uparrow{\scriptstyle \ell_{\mathrm{kl}}} \\ TB(X) & \overset{TB(\mathrm{kl}_c)}{\dashrightarrow} & TBT(\Psi) \end{array}$$

Logic and Steps

1. An adjunction $F \dashv G$ between categories $\mathbf{C} \leftrightarrows \mathbf{D}^{\mathrm{op}}$.
2. A functor T on $\mathbf{C}$ with a step $\tau \colon TG \Rightarrow G$.
3. A functor $B \colon \mathbf{C} \to \mathbf{C}$ and a functor $L \colon \mathbf{D} \to \mathbf{D}$ with a step $\delta \colon BG \Rightarrow GL$.
4. An initial algebra $\alpha \colon L(\Phi) \overset{\cong}{\Rightarrow} \Phi$.



$$\text{where } H = BT \text{ or } H = TB \qquad (1)$$

$$\begin{array}{ccc} X & \overset{\log_c}{\dashrightarrow} & G(\Phi) \\ {\scriptstyle c}\downarrow & & \uparrow{\scriptstyle \ell_{\log}} \\ TB(X) & \overset{TB(\log_c)}{\dashrightarrow} & TBG(\Phi) \end{array} \qquad \begin{array}{ccc} Y & \overset{\log_d}{\dashrightarrow} & G(\Phi) \\ {\scriptstyle d}\downarrow & & \uparrow{\scriptstyle \ell^{\log}} \\ BT(Y) & \overset{BT(\log_d)}{\dashrightarrow} & BTG(\Phi) \end{array}$$

Relating the approaches

In the paper: under certain assumptions, algebra morphisms of the form

$$
\begin{array}{ccc}
BT(\Theta) & \longrightarrow & BTG(\Phi) \\
{\scriptstyle \ell_{\mathrm{em}}}\big\downarrow & & \big\downarrow{\scriptstyle \ell^{\mathrm{log}}} \\
\Theta & \longrightarrow & G(\Phi)
\end{array}
\qquad
\begin{array}{ccc}
TBT(\Psi) & \longrightarrow & TBG(\Phi) \\
{\scriptstyle \ell_{\mathrm{kl}}}\big\downarrow & & \big\downarrow{\scriptstyle \ell_{\mathrm{log}}} \\
T(\Psi) & \longrightarrow & G(\Phi)
\end{array}
$$

relating different approaches to trace semantics (Logic/EM and Logic/Kleisli)

Conclusions

- Trace semantics captured through corecursive algebras
- These corecursive algebras are constructed from a final coalgebra on another category using an adjunction and a step
- Encompasses the main approaches to coalgebraic trace semantics
- Compare logical approaches to both Eilenberg-Moore and Kleisli approaches