Predicate Liftings and Functor Presentations in Coalgebraic Expression Languages

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Introduction

- Expression languages describe system behaviour:
  - Regular expressions
  - Kleene algebra with tests
  - Languages for LTS
- Coalgebraic expression languages
  - Silva et al. 2009a/b: Kripke polynomial functors
  - Silva et al. 2009c: Quantitative polynomial functors
  - Silva et al. 2010: Finitary set functors
  - Myers 2013: Finitary functors on varieties.

Introduction

- The μ-calculus contains an expression language for LTS
- Here: Generalize this to set coalgebras
- Use strongly expressive sets of singleton-preserving predicate liftings
- Conversions between predicate liftings and functor operations (Moss liftings)
- Kleene theorem

Characteristic Formulas in the μ-Calculus

Kripke frame

$$\begin{array}{ccc} x_1 & \rightleftarrows & x_2 \\ & & \downarrow \\ & & x_3 \end{array}$$

described by greatest fixpoint of

$$\begin{aligned} x_1 &= \Diamond x_2 \wedge \Box x_2 && = \nabla\{x_2\} \\ x_2 &= \Diamond x_1 \wedge \Diamond x_3 \wedge \Box(x_1 \vee x_3) && = \nabla\{x_1, x_3\} \\ x_3 &= \Box\bot && = \nabla\emptyset. \end{aligned}$$

(Graf/Sifakis 1986, Godskesen/Ingólfsdottir/Zeeberg 1987, Steffen/Ingólfsdottir 1994)

Functor Presentations

- Functor operation for T = natural transformation $X^n \to TX$
- Functor presentation of T = jointly surjective set of operations
- T is finitary iff it has a presentation

Functor Presentations – Examples

- *Finite powerset*: $(x_1, \dots, x_n) \mapsto \{x_1, \dots, x_n\}$
- *Finite distributions*: $(x_1, \dots, x_n) \mapsto p_1 x_1 + \dots + p_n x_n$
- *Finitary monotone neighbourhoods*:
  ($\mathcal{M}(X) = \{\mathfrak{A} \in \mathcal{Q}(\mathcal{Q}(X)) \mid \mathfrak{A}$ upclosed$\}$, $\mathcal{Q}$ contravariant powerset)

$$((x_{i1}, \dots, x_{ik_i})_{i=1,\dots,n}) \mapsto \{A \subseteq X \mid \exists i.\, \{x_{i1}, \dots, x_{ik_i}\} \subseteq A\} \in \mathcal{M}_\omega(X)$$

Predicate Liftings

…of arity n are natural transformations

$$\mathcal{Q}^n \to \mathcal{Q}T^{op}$$

where $\mathcal{Q}$ denotes contravariant powerset.

Standard examples:

- *Powerset*: $\Box_X(A) = \{B \in \mathcal{P}(X) \mid B \subseteq A\}$
  $\Diamond_X(A) = \{B \subseteq X \mid B \cap A \neq \emptyset\}$
- *Distributions*: $(L_p)_X(A) = \{\mu \in \mathcal{D}(X) \mid \mu(A) \geq p\}$
- *Monotone neighbourhoods*: $\Box_X(A) = \{\mathfrak{A} \mid A \in \mathfrak{A}\}$
  $\Diamond_X(A) = \{\mathfrak{A} \mid \forall B \in \mathfrak{A}.\, A \cap B \neq \emptyset\}$.

Singleton-preserving Predicate Liftings

λ/n preserves singletons if, well,

$$|\lambda(\{x_1\},\dots,\{x_n\})| = 1$$

for all $x_1,\dots,x_n$.

Example:

$$\lambda_X^n(A_1,\dots,A_n) = \{B \in \mathcal{P}(X) \mid B \subseteq A_1 \cup \dots \cup A_n, \forall i.\, A_i \cap B \neq \emptyset\}$$

– then

$$\lambda_X^n(\{x_1\},\dots,\{x_n\}) = \{x_1,\dots,x_n\}$$

Strong Expressivity

Set Λ of predicate liftings strongly expressive $\iff$

$$\forall t \in TX \, \exists \lambda/n \in \Lambda, x_1, \dots, x_n \in X. \{t\} = \lambda_X(\{x_1\}, \dots, \{x_n\}).$$

(stronger than separation)

E.g. $\Lambda = \{\lambda^n \mid n \in \mathbb{N}\}$ as above.

Functor Operations From Predicate Liftings

Given a monotone singleton-preserving λ/n,

$$\tau^\lambda(x_1,\ldots,x_n) \in \lambda(\{x_1\},\ldots,\{x_n\})$$

defines a functor operation.

$\rightarrow$ strongly expressive sets of monotone singleton-preserving predicate liftings induce functor presentations.

Lax Extensions

(Marti/Venema 2012, 2015)

…map $R \subseteq X \times Y$ to

$$LR \subseteq TX \times TY$$

subject to monotonicity, lax preservation of composition, $Tf \subseteq Lf$

Diagonal-preserving lax extensions capture behavioural equivalence; e.g. Barr extension if T preserves weak pullbacks.

Lax nabla

$$\begin{aligned} \nabla^L_X : T\mathcal{Q}X &\to \mathcal{Q}T^{op}X \\ \Phi &\mapsto \{t \in TX \mid t\ L(\in_X)\ \Phi\} \end{aligned}$$

Moss Liftings

From operation $\tau : (-)^n \to T$ obtain Moss lifting
(Kurz/Leal 2009, Marti/Venema 2012, 2015)

$$\lambda^\tau = (\mathcal{Q}^n \overset{\tau\mathcal{Q}}{\Longrightarrow} T\mathcal{Q} \overset{\nabla^L}{\Longrightarrow} \mathcal{Q}T^{op}),$$

that is,

$$\lambda^\tau_X(X_1, \dots X_n) = \{t \in TX \mid t \; L(\in_X) \; \tau_{\mathcal{Q}X}(X_1, \dots, X_n)\}.$$

E.g. $T = \mathcal{P}_\omega$, $\tau(x_1, \dots, x_n) = \{x_1, \dots, x_n\}$, $L =$ Barr extension yields

$$\lambda^\tau(A_1, \dots, A_n) = \{B \in \mathcal{P}_\omega(X) \mid B \subseteq A_1 \cup \dots \cup A_n, \forall i.\, B \cap A_i \neq \emptyset\}.$$

Theorem Moss liftings (for a presentation) preserve singletons and are strongly expressive

Example: Monotone Neighbourhoods

Above presentation + standard lax extension induce

$$\lambda_X((A_{ij})) = \bigcap_i \Box_X(\bigcup_j A_{ij}) \cap \bigcap_\pi \Diamond_X(\bigcup_i A_{i\pi(i)})$$

where π ranges over functions such that $\pi(i) \in \{1, \ldots, k_i\}$ for all i

An Expression Language

- Fix set $\mathcal{L}$ of finitary modalities L, interpreted by predicate liftings $[\![L]\!]$
- $\phi ::= z \mid \nu z.\,\phi \mid L(\phi_1, \dots \phi_n) \qquad (z \in \mathsf{V}, L/n \in \mathcal{L})$.
- Require the $[\![L]\!]$ to be singleton-preserving and strongly expressive
- Given T-coalgebra $C = (X, \xi)$, valuation $\kappa : \mathsf{V} \to \mathcal{P}(X)$: extension $[\![\phi]\!]^\kappa_C \subseteq X$,

$$[\![L(\phi_1, \dots, \phi_n)]\!]^\kappa_C = \xi^{-1}[[\![L]\!]_X([\![\phi_1]\!]^\kappa_C, \dots, [\![\phi_n]\!]^\kappa_C)]$$

- Fragment of the coalgebraic μ-calculus

A Kleene Theorem

Theorem Closed and guarded expressions ϕ correspond to states in finite coalgebras, i.e.

- All states satisfying ϕ are behaviourally equivalent.
- Every ϕ has a finite model.
- Every state in a finite coalgebra satisfies some ϕ.

A Canonical Model

Coalgebra structure ε on $\mathcal{E}_0 :=$ closed and guarded expressions:

$$\varepsilon(L(\phi_1,\dots,\phi_n)) \in [\![L]\!](\{\phi_1\},\dots,\{\phi_n\})$$
$$\varepsilon(\nu x.\phi) = \varepsilon(\phi[\nu x.\phi/x]).$$

Then $(\mathcal{E}_0, \varepsilon)$ is locally finite, and

Theorem

$$x \in [\![\phi]\!]_C \quad \text{iff} \quad x \text{ is behaviourally equivalent to } \phi$$

(Similar constructions *define* the semantics
in previous coalgebraic languages.)

Conclusions

- Expression language for set coalgebras
- Fragment of the coalgebraic μ-calculus
- Abstractly:
  Strongly expressive sets of singleton-preserving predicate liftings
- Concretely:
  Functor presentation + diagonal-preserving lax extension
- Kleene theorem, canonical model construction
- Future work:
  - Extend to algebraic base categories
  - Coarser equivalences, maybe via graded monads