Fibrational Bisimulations and Quantitative Reasoning

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David Sprunger*, Shin-ya Katsumata,
Jérémy Dubut, and Ichiro Hasuo

National Institute of Informatics, Japan

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Showing relationships between invariants of different “types”

Examples

Language equivalence sends a DFA (I, τ) to a relation

$$LE : (I, \tau) \mapsto \{(x, y) \mid L(x) = L(y)\}$$

Language inclusion sends a DFA to a relation

$$LI : (I, \tau) \mapsto \{(x, y) \mid L(x) \subseteq L(y)\}$$

relationship: $LE(I, \tau) \subseteq LI(I, \tau)$

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relationship: $LE(I, \tau) \subseteq LI(I, \tau)$

Counting distance sends a DFA to a[n extended] metric

$$CD : (I, \tau) \mapsto \lambda xy.\#(L(x) \Delta L(y))$$

relationship: $LE(I, \tau) = CD(I, \tau)^{-1}(\{0\})$

Showing relationships between invariants of different "types"

	Examples	Abstraction
Types	relations, metrics, topologies...	fibrations
Invariants	DFAs $\mapsto$ bisimilarity WAs $\mapsto$ behavioural metric	liftings
Relationships	"metric has bisimilarity at its kernel"	???

Previous work

Idea (Hermida & Jacobs '98,
and Hasuo, Cho, Kataoka, Jacobs '13):
Given a functor F and a fibration, every lifting of F induces a coalgebraic predicate [invariant] on each F-coalgebra.

The Catch: You do not define the invariant directly with a formula—the lifting induces it.

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The Catch: You do not define the invariant directly with a formula—the lifting induces it.

Project: How many liftings can we find? Which liftings induce interesting invariants?

Problem: How can we prove relationships between invariants **strictly categorically**?

Talk structure

1. Problem statement
2. Review of fibrations and coalgebraic invariants
3. Showing relationships between invariants
4. Ways to lift functors along fibrations

Reviewing coalgebraic invariants

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$\mathbf{CLat}_\wedge$-fibrations, technically

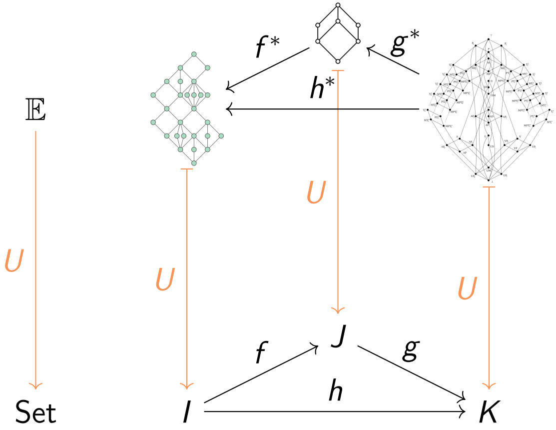
A *fibration* (over Set) is a functor $U : \mathbb{E} \to \mathrm{Set}$ such that every function $f : I \to UY$ has a Cartesian lifting $\overline{f} : f^*Y \to Y$.

A $\mathbf{CLat}_\wedge$-fibration is a fibration such that

- each fiber category is a complete lattice
- reindexing preserves meets in fibers

These correspond to functors $\mathrm{Set}^{op} \to \mathbf{CLat}_\wedge$ via the Grothendieck construction, where $\mathbf{CLat}_\wedge$ is the category of complete lattices and meet-preserving functions.

$\mathbf{CLat}_\wedge$-fibrations, generally



- "Fiber over I" is a complete lattice
- f^* reindexes (takes preimage)

$\mathbf{CLat}_\wedge$-fibrations, examp-ally

The obvious forgetful functor from each of the following categories is a $\mathbf{CLat}_\wedge$-fibration:

- **Pre** is preorders and monotone functions.
- **EnRel** is endorelations and relation-preserving functions.
- $\mathbf{PMet}_b$ is b-bounded pseudometric spaces and non-expansive functions, $b \in [0, \infty]$.
- **BVal** is all binary $[0, \infty]$-valuations ($r : I \times I \to [0, \infty]$) and non-expansive functions.
- **Top** is topological spaces and continuous functions.
- **Meas** is measure space and measurable functions.

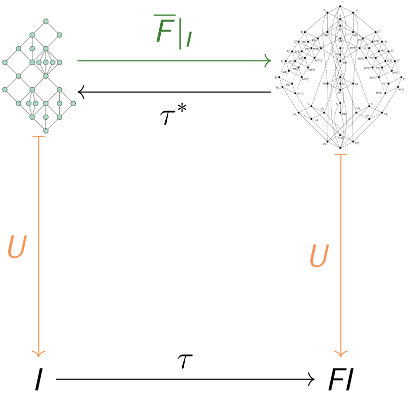
Liftings

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\overline{F}} & \mathbb{E} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathrm{Set} & \xrightarrow[F]{} & \mathrm{Set} \end{array}$$

A lifting of $F : \mathrm{Set} \to \mathrm{Set}$ along a fibration $U : \mathbb{E} \to \mathbb{E}$ is a functor $\overline{F} : \mathbb{E} \to \mathbb{E}$ such that $\overline{F} \circ U = U \circ F$.

We usually denote a lifting by $\overline{F}$, but sometimes also include the total category $(\overline{F}, \mathbb{E})$.

Making invariants



[Hermida & Jacobs]
We are given an F-coalgebra (I, τ), a fibration U and a lifting $\overline{F}$ along that fibration.

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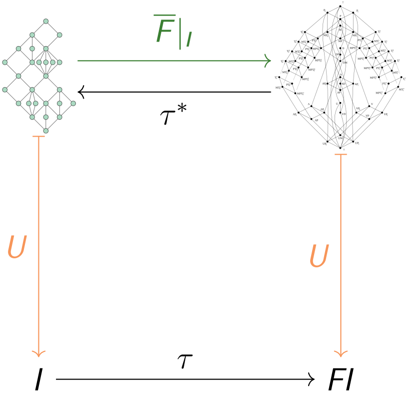

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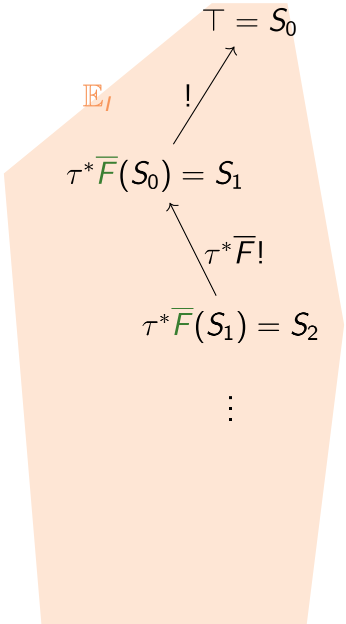
Functors between fiber categories are monotone functions between lattices.

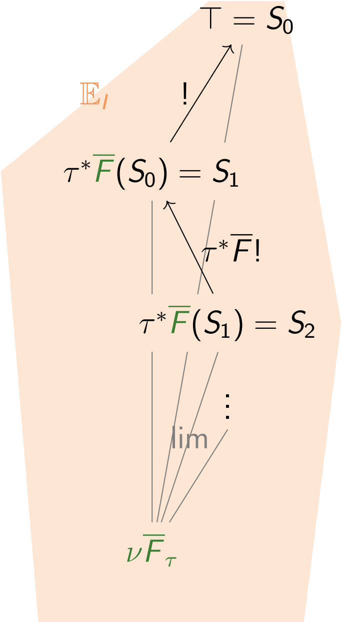
- $\overline{F}|_I : \mathbb{E}_I \to \mathbb{E}_{FI}$ is one
- $\tau^* : \mathbb{E}_{FI} \to \mathbb{E}_I$ is another

The greatest fixed point of the composite functor $\tau^* \circ \overline{F}|_I$ is the invariant induced by $\overline{F}$.

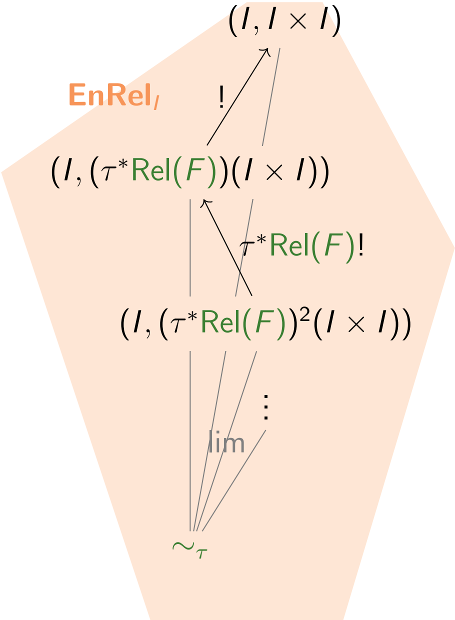
Making invariants



Making invariants



Making invariants: bisimilarity



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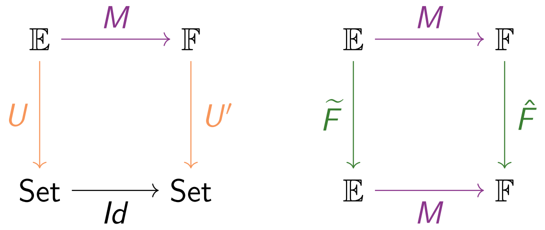
Contribution 1: relationships between invariants

Idea: Upgrade the *collection* of liftings to a *category* by adding endolifting morphisms.

Definition: An *endolifting morphism* M from a lifting $(\widetilde{F}, \mathbb{E})$ to a lifting $(\hat{F}, \mathbb{F})$ makes the following diagrams commute:



Namely,
$U' \circ M = U$, and
$\hat{F} \circ M = M \circ \widetilde{F}$.

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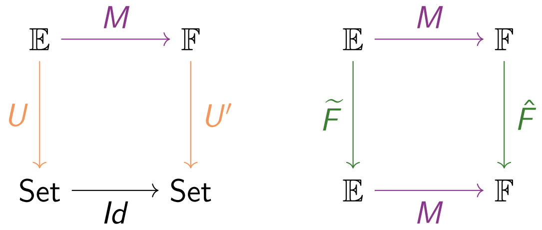

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$U' \circ M = U$, and
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Hope: Good M's should send $\nu \widetilde{F}_\tau$ to $\nu \hat{F}_\tau$.

Endolifting morphism example

For this slide, let $FI = 2 \times I \times I$.

- The **EnRel** lifting $\mathrm{Rel}(F)$ induces bisimilarity.
- The following **BVal** lifting induces the counting distance metric on DFAs:

$$\widetilde{F}(I, d) = (FI, \lambda bijb'i'j'. [|b - b'| + d(i, i') + d(j, j')])$$

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The truncation functor $T_0 : \mathbf{BVal} \to \mathbf{EnRel}$ is defined by $(I, d) \mapsto (I, \{(i, j) : d(i, j) = 0\})$. It is an endolifting morphism from $\widetilde{F}$ to $\mathrm{Rel}(F)$.

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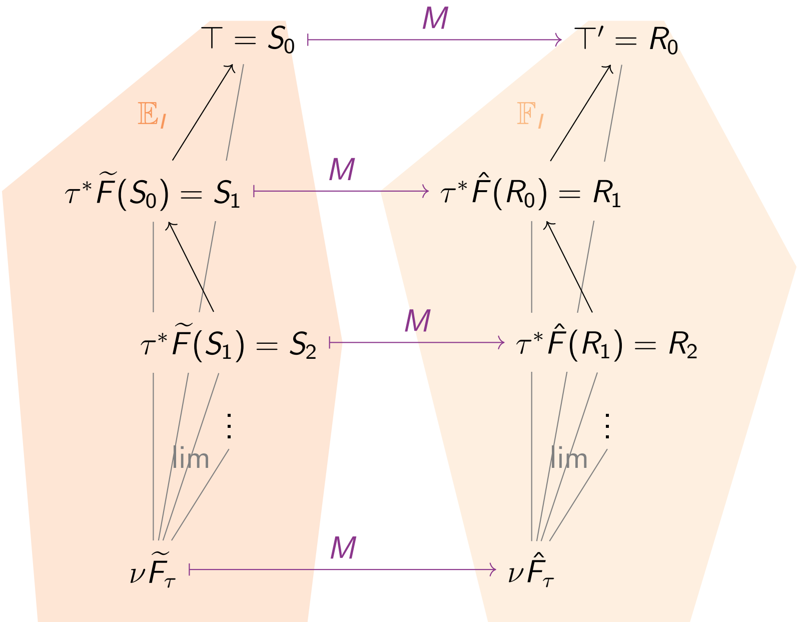
Indeed, we want to confirm that

$$T_0 \nu \widetilde{F}_\tau = \{(i, j) : CD(I, \tau)(i, j) = 0\} = \sim_\tau = \nu \, \mathrm{Rel}(F)_\tau$$

Matching up final sequences



Matching up final sequences

To make this sequence argument work, we need $M\top = \top'$, $M\tau^*\widetilde{F} = \tau^*\hat{F}M$, and M to preserve limits.

Theorem

If M is an endolifting morphism which is also a fibration map and preserves fibred meets, then it also preserves invariants.

Proof: "Fibration map" means $M\tau^* = \tau^* M$, and endolifting morphism gives $M\widetilde{F} = \hat{F}M$—these form the successor step. Preserving fibred meets is the initial and limit step.

Example consequences

From this theorem, you can derive:

- The counting distance assigns distance 0 to exactly the bisimilar states.
- There are liftings of every polynomial Set functor to **BVal** whose invariants assign distance 0 to exactly the bisimilar states.
- Certain liftings to **Top** induce a topology where topological indistinguishability is exactly bisimilarity.
- ...

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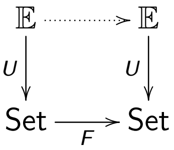
Contribution 2: the quest for liftings

We have two methods for creating liftings, using enriched left Kan extensions and codensity (right Kan extensions). These were inspired by existent liftings from the literature.

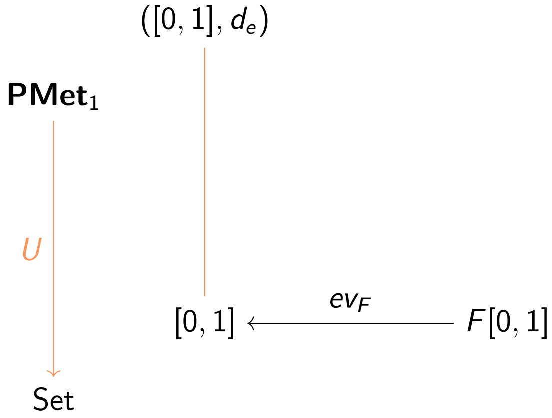
In general, we are considering the following situation:



and trying to fill in the dotted arrow.

Kantorovich lifting

(From Baldan, Bonchi, Kerstan, König ‘14)



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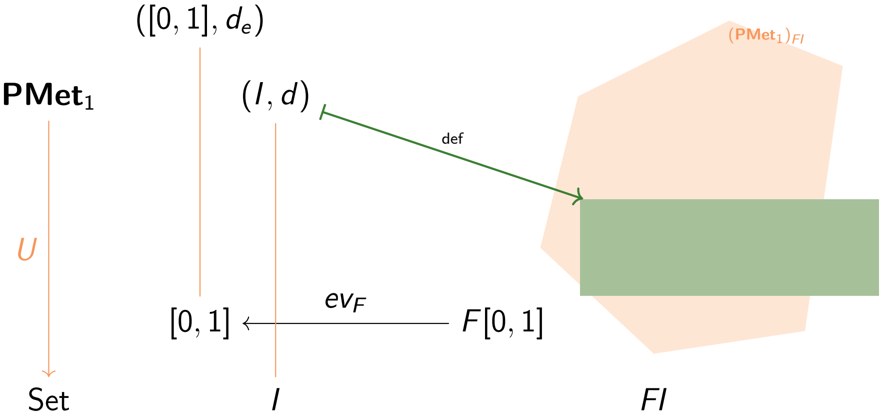

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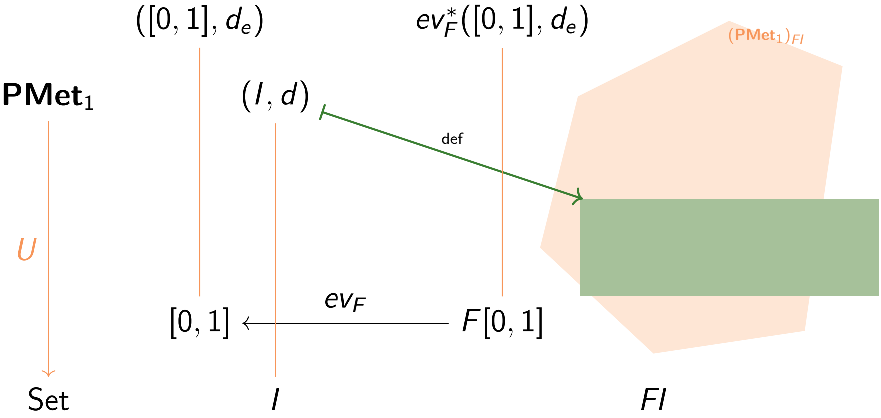

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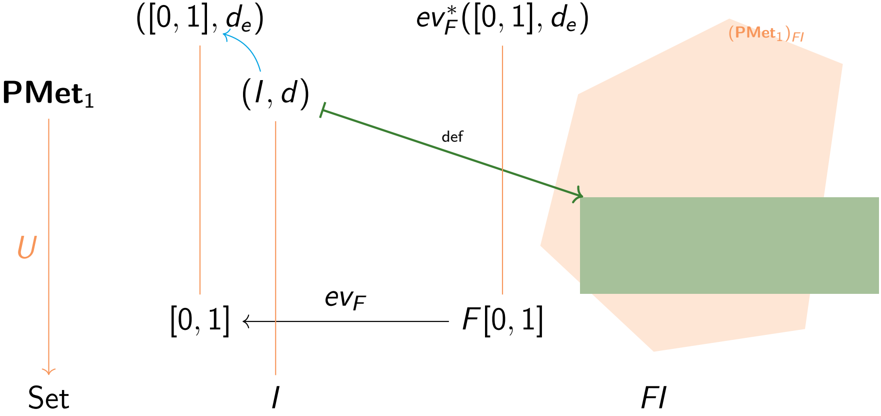

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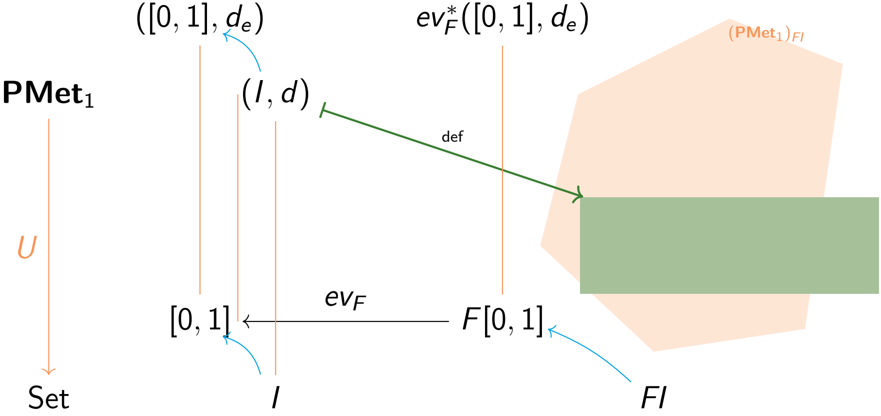

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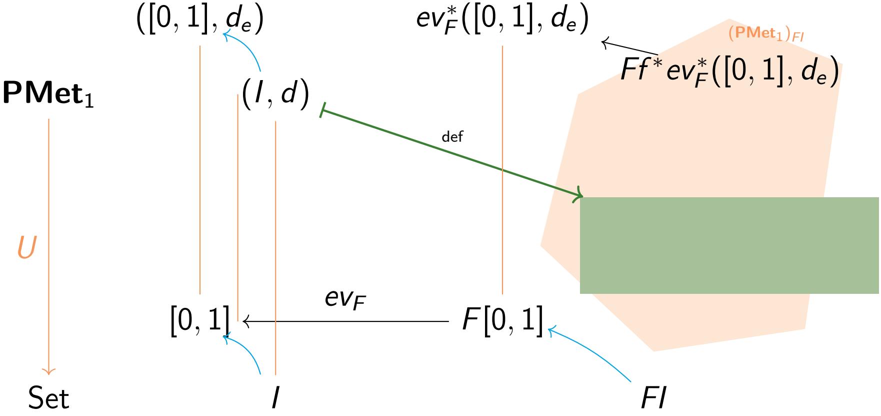

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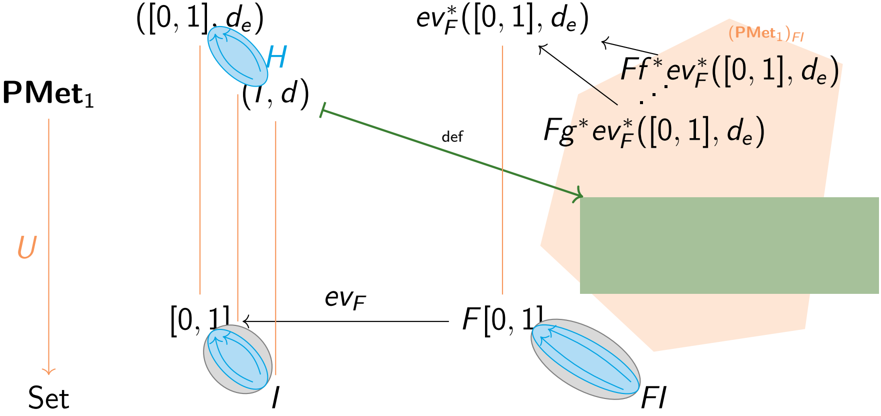

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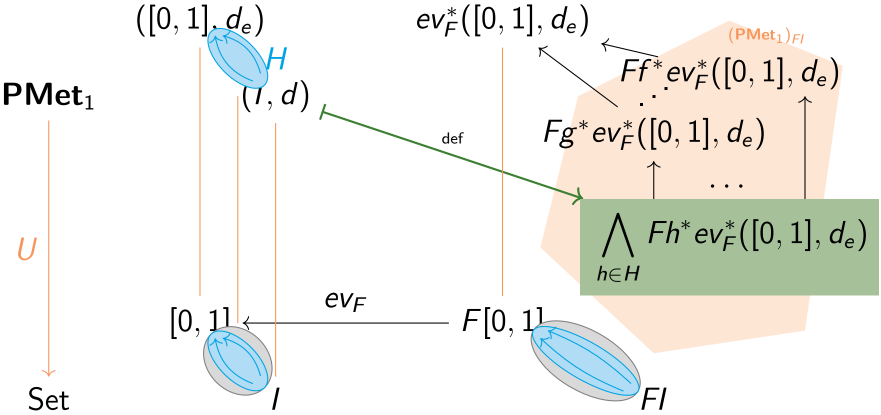
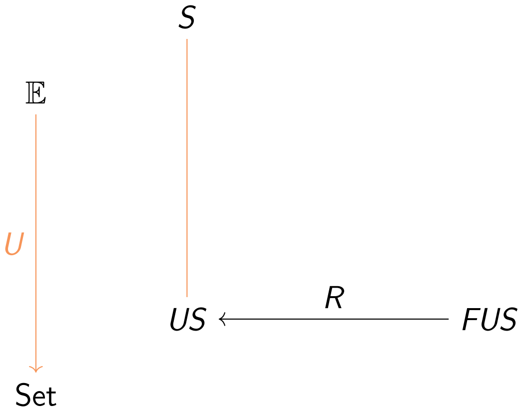


Generalized Kantorovich lifting



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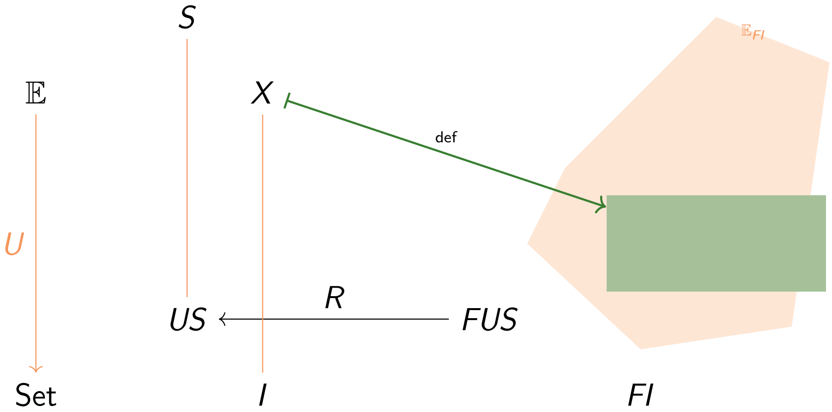

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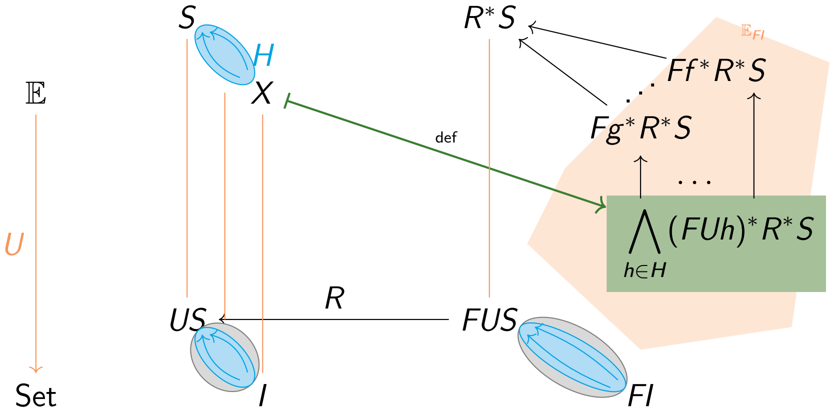


Codensity lifting

Our codensity lifting generalizes this a bit further by allowing a **set** of reference objects.

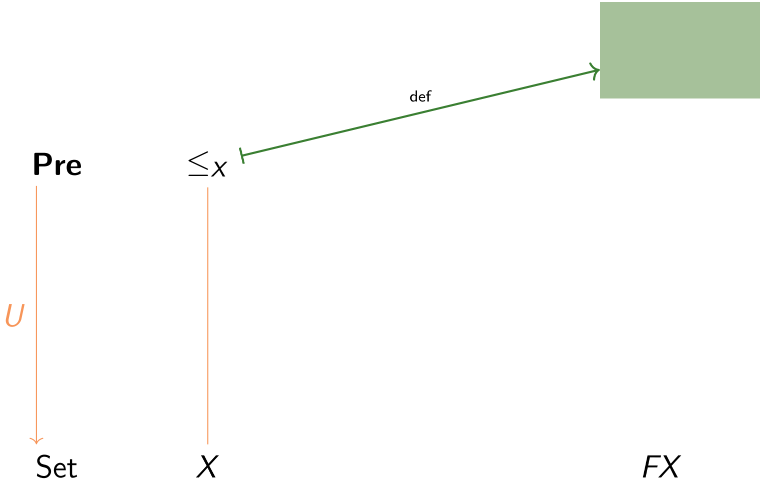
Theorem

Given a $\mathbf{CLat}_\wedge$-fibration, a Set endofunctor F, and parameters S and R, the codensity lifting just described is a lifting of F along the fibration.

This was also inspired by the codensity lifting of monads. (Katsumata & Sato, '15)

Preorder lifting

(From Balan, Kurz, Velebil ‘15)



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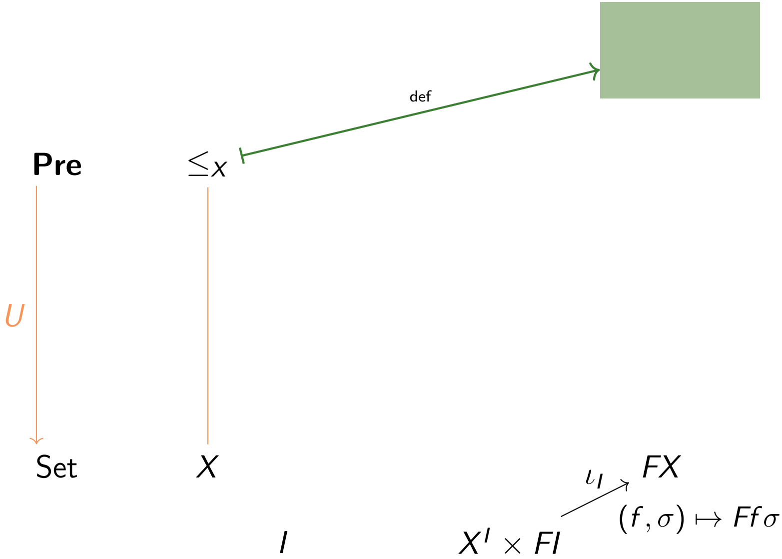

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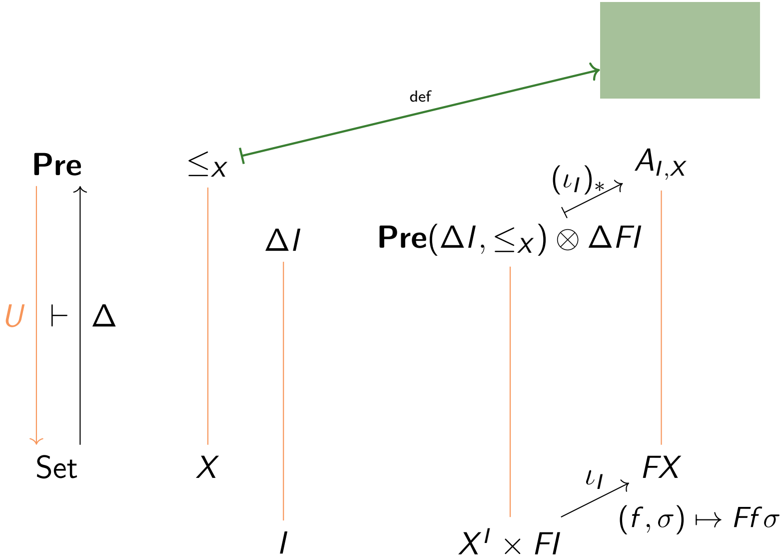

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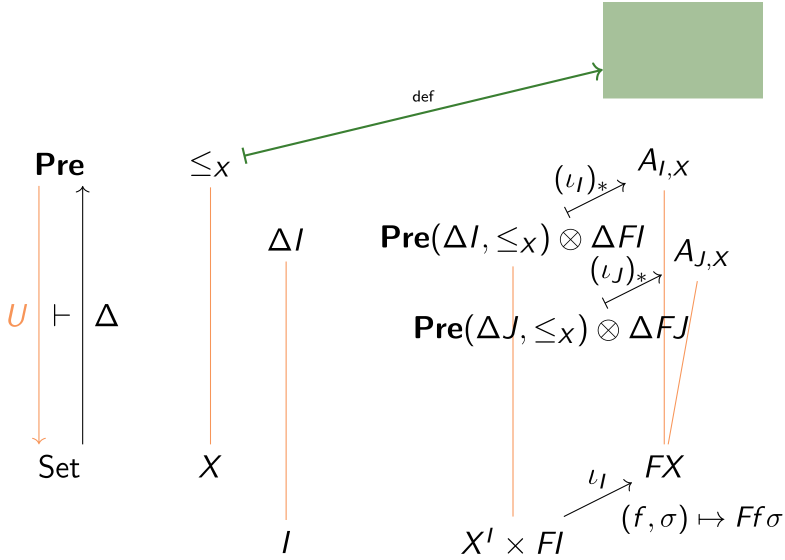

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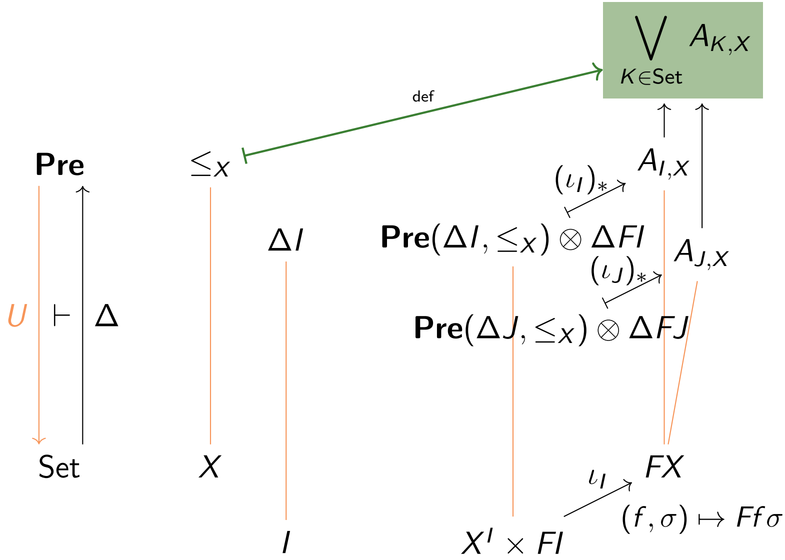
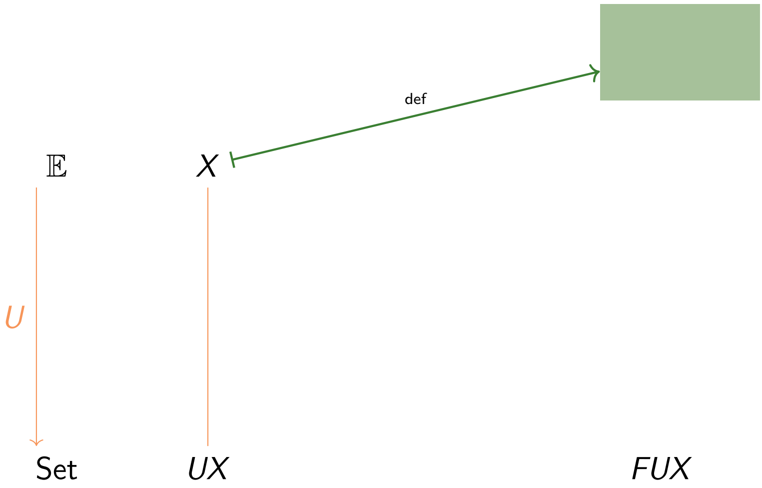


Generalized preorder lifting



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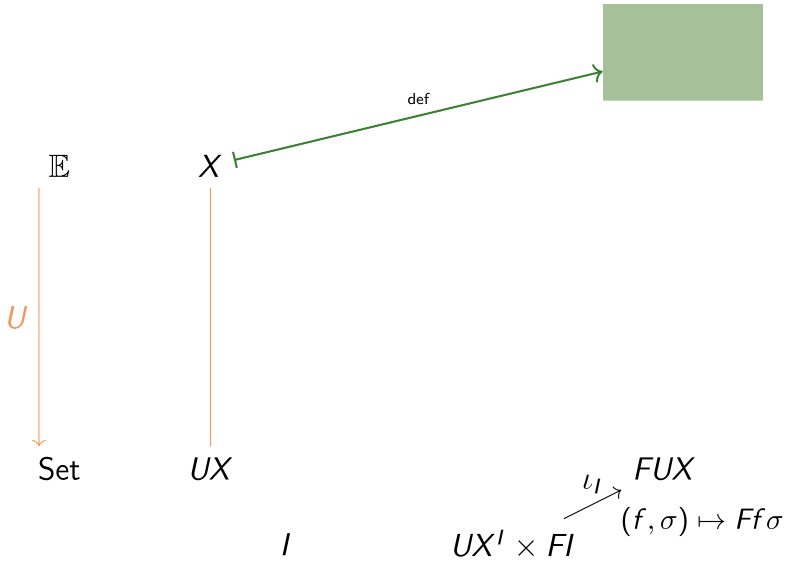

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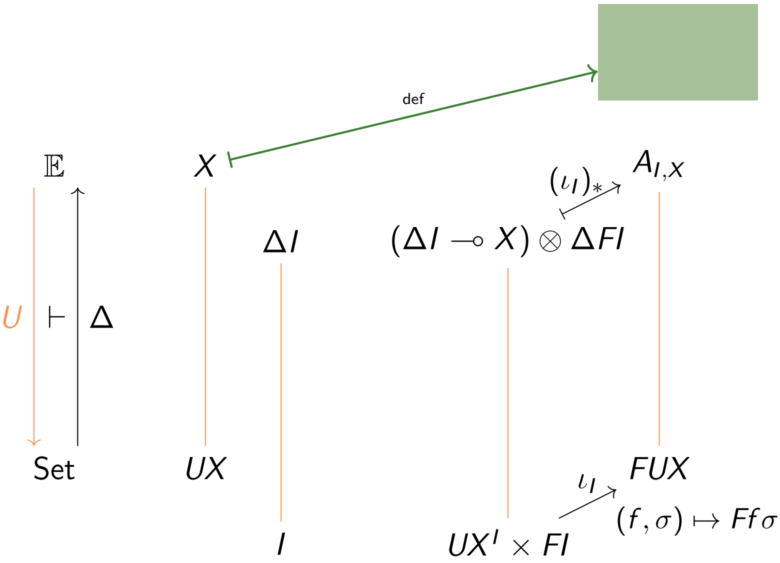

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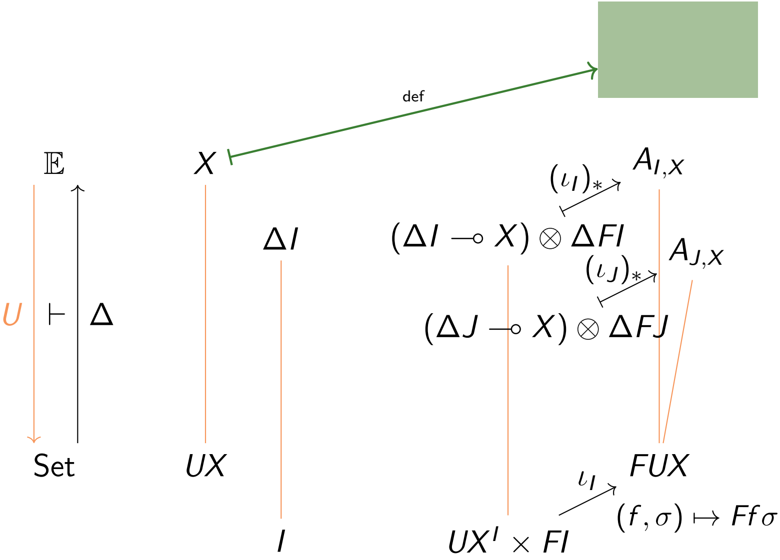

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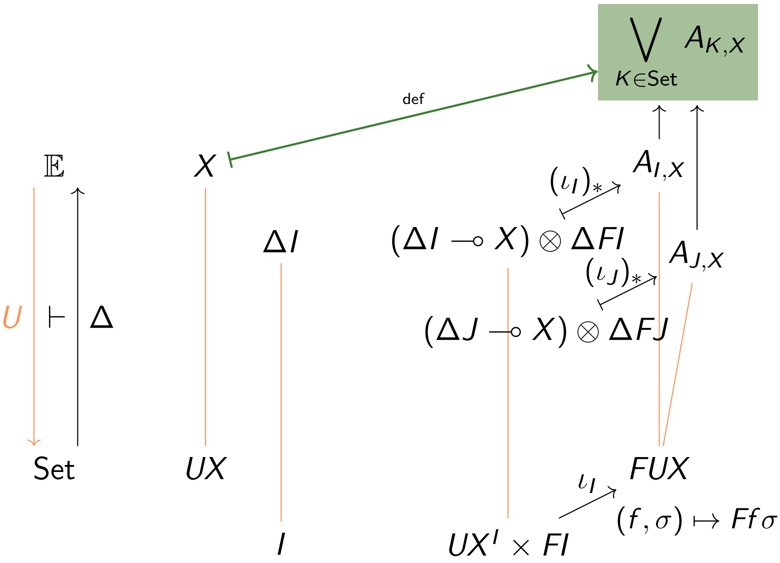

Lifting by enriched Lan extensions

All $\mathbf{CLat}_\wedge$-fibrations have the structure we need to make this construction work (bifibration, symmetric monoidal closed structure, s.m. left adjoint to U). (Kelly & Rossi, '85)

We can actually generalize a bit further and replace ΔF with an arbitrary extension C of F along U.

Theorem

Given a $\mathbf{CLat}_\wedge$-fibration, a functor F and an extension C of F, the construction just described gives a lifting of F along the fibration.

Summary

New recipes for liftings
in a variety of fibrations.

New criteria for verifying relationships
between the induced invariants.

Thanks!