Causality, Quantum Channels, and Mealy Machines

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(with Juriaan Rot & Henning Basold)
CAUSALITY appears in many forms:

In Physics:

In statistics:

In computation:

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THE Goal: unify the way we formulate and reason about causal relationships (between processes, events, programs, ….)

THE Tool: category theory (obviously :-)

in particular, using

**PRECAUSAL CATEGORIES**

(:= SMC's + extra structure)
Symmetric Monoidal Categories:

* two compositions \( \circ \) and \( \otimes \)
  
  sequential \hspace{4cm} parallel

* swap maps \( \sigma_{A,B} : A \otimes B \rightarrow B \otimes A \)

* + axioms...

**String Diagram Notation:**

\( f : A \rightarrow B \Rightarrow \)

\[ \begin{array}{c}
|B \\
\hline
f
\hline
|A
\end{array} \]

\( g \circ f \Rightarrow \)

\[ \begin{array}{c}
|C \\
\hline
f
\hline
|B
\end{array} \]

\( f \otimes g \Rightarrow \)

\[ \begin{array}{c}
|A' \\
\hline
f
\hline
|B'
\end{array} \]

\( \sigma_{A,B} \Rightarrow \)

\[ \begin{array}{c}
|B \\
\hline
A
\end{array} \quad \begin{array}{c}
|A \\
\hline
B
\end{array} \]

\( 1_A \Rightarrow \)

\[ \begin{array}{c}
|A
\end{array} \]

\( (\text{or } 1_I) \Rightarrow \)

\[ \begin{array}{c}
|I
\end{array} \]

\[ \begin{array}{c}
|I
\end{array} \]
SMC axioms $\iff$ "only connectivity matters"

\[
\begin{align*}
\sigma_{D,H \otimes 1_{F \circ G}} \circ (1_D \otimes g \otimes 1_{F \circ G}) \circ (f \otimes h) \circ (\chi \otimes 1_{B \otimes C}) &= \text{SMC} \\
(g \otimes 1_{E \circ F \circ G}) \circ (\sigma_{D,E} \otimes 1_{F,G}) \circ (1_A \otimes \sigma_{F,G,B}) \circ (1_A \otimes h \otimes 1_B) \circ (\chi \otimes \sigma_{B, C})
\end{align*}
\]
Q: how do I get some causality into my SMC?

A: introduce a new process $d_A : A \to I$

$$d_A := \frac{\cdot}{TA}$$

for discarding.

DEF A process is causal if it satisfies:

$$\frac{\frac{\cdot}{TB}}{\frac{\cdot}{TA}} = \frac{\cdot}{TA}$$
Example:

\[ \text{Mat}(IR_+) \begin{cases} \forall n \in \mathbb{N} \\ \forall m \in \mathbb{N} \\ \otimes \text{ tensor product: } Z_{i,j,k} = X_{i,j} Y_{j,k} \end{cases} \]

\[ \begin{array}{c} \tilde{1}_k := \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \\ \boxed{\times} = \tilde{1} \Rightarrow X \text{ is a stochastic matrix.} \end{array} \]
OTHER EXAMPLES:

* all functions (or morphisms in a category where I is terminal) are causal.

* in Rel causal means total rules out "assertion" type processes

* in CPM, causal morphisms are quantum channels

\[ \text{map} : \text{Hilbert spaces } H, K \]

\[ \text{arr: completely positive maps } L(H) \rightarrow L(K). \]

\[ \otimes : \text{tensor product} \]
A function \( f : A^\omega \to B^\omega \) is causal if for all \( n \in \mathbb{N} \), \( \sigma \in A^\omega \):

\[
\exists f' : A^n \to B^n. \quad f'(\sigma)|_n = \hat{f}(\sigma|_n)
\]

"The \( n \)-th output only depends on the first \( n \) inputs."
Q: can we picture "your" causality?

\[ \forall n \in \mathbb{N}. \exists \text{iso } A^\omega \cong A^n \times A^\omega \]

**Restriction:**

\[ \begin{array}{c}
\downarrow A^\omega \\
\sigma \\
\end{array} \cong \begin{array}{c}
\downarrow A^n \\
\downarrow (A^\omega)_n \\
\sigma \\
\end{array} \quad \sigma \big|_n \leftrightarrow \begin{array}{c}
\downarrow A^n \\
\sigma \\
\end{array} \cong \begin{array}{c}
\downarrow A^\omega \\
\sigma \\
\end{array} \]
\[ \forall f', \frac{f'(\sigma)}{A^n} = f'(\sigma|_n) \implies \frac{f}{\sigma} = \frac{f'}{\sigma} \]

Since this holds for all \( \sigma \), we can write:

\[ \forall n. \exists f : A^n \to B^n. \frac{B^n}{B^n} = \frac{f}{\sigma} = \frac{f'}{A^n} \]

This eq has a name in Foundations of Physics, it's called one-way non-signalling.
An important theorem in quantum information theory (and the most important axiom for precausal categories...)

\[
\text{One-way Non-signalling} \quad \Rightarrow \quad \text{Semi-localisable}
\]

\[
\text{If } \quad \left| \begin{array}{c} f \end{array} \right| = \left| \begin{array}{c} f \end{array} \right| = \frac{1}{2}
\]

\[
\text{If } f, f_1 \quad \left| \begin{array}{c} f \end{array} \right| = \left| \begin{array}{c} f_1 \end{array} \right|
\]

* Classical case: Follows from "Conditional Disintegration"
* Quantum case: Eggeling, Schlingemann, Werner '02
Does this correspond to something for causal stream functions?

First massage a bit:

\[
\forall n \in \mathbb{N} . \exists f_1, f_2 . \begin{array}{c}
B^n \\
A^n
\end{array} \begin{array}{c}
B^\omega \\
A^\omega
\end{array} = \begin{array}{c}
B^n \\
A^n
\end{array} \begin{array}{c}
\{ f_2 \} \\
A^\omega
\end{array}
\]

\[
\text{SEMI-LOCALISABLE}
\]

\[f \text{ is } \omega\text{-SEMI-LOC.} \]

:= \left( \exists f_1, f_2 . \begin{array}{c}
B^n \\
A^n
\end{array} \begin{array}{c}
B^\omega \\
A^\omega
\end{array} = \begin{array}{c}
B^n \\
A^n
\end{array} \begin{array}{c}
\{ f_2 \} \\
A^\omega
\end{array} + \forall x . \begin{array}{c}
B^\omega \\
A^\omega
\end{array} \begin{array}{c}
\{ f_2 \} \\
\omega\text{-SEMI-LOC.}
\end{array} \right)\]
MEALY MACHINES := coalgebras of

\[ M_{A \to B}(-) := (B \times -)^A \]

\[ \rightarrow \text{(det.) machine that implements a causal} \]

\[ \text{stream function:} \]

\[ \begin{array}{c}
0 \rightarrow \bullet \leftarrow 0 \leftarrow 1 \rightarrow \bullet \\
0 \rightarrow \bullet \leftarrow 0 \rightarrow 1 \\
\end{array} \]

\[ \begin{array}{c}
000101 \rightarrow f \rightarrow 011101 \ldots \\
\end{array} \]
Theorem: Causal Functions are final co-algebra of $K(-)$.

For $f = f_1 \oplus f_2$,

\[ \phi: f \rightarrow (f_1, f_2) \]

is (up to currying) the structure map.

Concretely: this is synthesis of Mealy machines from causal functions using stream derivatives.