Alternating Automata via Weak Distributive Laws

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Abstract
We make use of a weak notion of distributive law to take a second look at the coalgebraic modeling of alternating automata, especially determinization.

In a recent paper from Garner [4] interesting insights are brought in Beck’s theory of distributive laws [1]. In order to see the Vietoris monad as a lifting of the powerset monad, Garner is led to make use of a notion of weak distributive law already stated in [3].

Definition 1 (Weak distributive law). A weak distributive law of a monad $S = (S, \nu, \omega)$ over a monad $T = (T, \eta, \mu)$ is a natural transformation $\delta : TS \Rightarrow ST$ such that $\delta \circ \mu_S = S\mu \circ \delta T \circ T\delta$, $\delta \circ T\omega = \omega T \circ S\delta \circ \delta S$, and $\delta \circ T\eta = \nu T$.

Together with this weak variant of distributive laws come weak notions of liftings and extensions of monads [4], along with a bijective correspondence between weak distributive laws, weak extensions, and (whenever idempotents split in the base category) weak liftings.

Definition 2 (Weak lifting). A weak lifting of $S$ to $EM(T)$ is a monad $\tilde{S}$ on $EM(T)$ along with two natural transformations $\pi : SU^\tilde{T} \Rightarrow U^\tilde{T}\tilde{S}$, $\iota : U^\tilde{T}\tilde{S} \Rightarrow SU^\tilde{T}$ such that $\pi \circ \iota = 1$ and the following diagrams commute.

Such a weak lifting yields a monad $\tilde{S}T = U^\tilde{T}\tilde{F}T$ on the base category. Examples include identifying the Vietoris monad [4] (resp. the convex powerset monad [5]) as the weak lifting of the powerset monad with respect to the ultrafilter monad (resp. the finite distribution monad).

Alternating automata can be defined as coalgebras for the $Set$-endofunctor $2 \times (PP^-)^A$, where $A$ is an alphabet. The semantics of alternating automata is guided by the following interpretation: an element $A \in PPX$ is seen as a disjunctive normal form $\bigvee_{U \in A} \bigwedge_{y \in U} y$, where there
is an arbitrary number of clauses of arbitrary length. In concrete terms, given an alternating
automaton \( \langle a, N \rangle : X \to 2 \times (PPX)^A \):
\[
\llbracket x \rrbracket (x) = a(x) \\
\llbracket x \rrbracket (aw) = \bigvee_{u \in N(x)(a)} \bigwedge \llbracket y \rrbracket (w)
\]
(3)

This modelling opens the chase of a possible distributive law of \( P \) over itself. As proved recently
[6], this does not exist, and actually there is no possible monad structure on \( PP \) at all; however
it still is possible to model alternating automata coalgebraically, e.g. by looking into Poset [2].

Coming back to our weak framework, Garner points out that there is a weak distributive law \( \delta \)
of the finite powerset monad \( P_f \) over \( P \) given by
\[
\delta_X(A) = \{ B \subseteq X \text{ finite} \mid B \subseteq \bigcup A \text{ and } \forall A \in A, A \cap B \neq \emptyset \}
\]
(4)

This paves the way for a new modelling of alternating automata as coalgebras for the functor
\( 2 \times (P_f)^-A \) — modelling that coincides with the usual one for finite systems. Let \( G = 2 \times (\cdot)^A \).

This modelling yields determinization for alternating automata as in the following diagram:

\[
\begin{array}{ccc}
\text{Coalg}(GPP_f) & \xrightarrow{F_f^G} & \text{Coalg}(\bar{G}\bar{P}) \\
\downarrow & & \downarrow \circ \delta_f \\
\text{Set} & \xrightarrow{F_f^G} & \text{EM}(P_f) \\
\end{array}
\]

where \( \bar{G} \) is the lifting of \( G \) to \( EM(P_f) \) arising from the known monad-functor distributive
law \( \lambda_X(S) = \langle \wedge (b, f) \in S, b, a \to \{ f(a) \mid (b, f) \in S \} \rangle \). The lifted \( F_f^G \) consists in transforming a
coalgebra \( c : X \to GPP_fX \) into \( X \xrightarrow{c} GPP_fX \xrightarrow{G_{\bar{P}f}} GU_{\bar{P}f}\bar{P}f^G \bar{P}f^G \bar{P}f^G X \) and
then taking the adjoint transpose \( c^\# : F_f^G X \xrightarrow{\bar{G}\bar{P}f} \bar{G}\bar{P}f^G X \). The lifted \( U_{\bar{P}f} \) maps \( c : (X, x) \to \bar{G}\bar{P}f(X, x) \)
to \( U_{\bar{P}f}(X, x) \xrightarrow{\bar{G}\bar{P}f^G} \bar{G}\bar{P}f(X, x) \Rightarrow \bar{G}\bar{P}f(\bar{G}f^G, x) \Rightarrow \ PU_{\bar{P}f}(X, x) \).

Further research includes considering semantics arising from such determinizations, looking at
potential weak distributive laws from \( P \) over itself, and investigating compositionality of
weak distributive laws in order to model more complex systems.

References

weak distributive laws. Submitted.