Discrete density comonads and graph parameters

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Resource-indexed comonads and combinatorial parameters

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Relating Structure to Power

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Capturing logic fragments and the equivalences on structures they induce in syntax-free form by comonads indexed by resource parameters:

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An unexpected and striking feature of this work:

Coalgebras $A \to C_k A$ for these comonads correspond to various natural forms of *resource-bounded tree decompositions* of structures

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We say that a comonad $\mathbb{C}$ *classifies a class* Δ if a finite structure A is in the class Δ precisely when A admits a $\mathbb{C}$-coalgebra. For example, the Ehrenfeucht–Fraïssé comonad $\mathbb{E}_k$ classifies the structures of tree-depth $\leq k$ and, similarly, the pebbling comonad $\mathbb{P}_k$ classifies tree-width $<k$.

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Thus we are led to ask:

When is a class of structures Δ classified by some comonad?

Main Theorem I

Note that two necessary conditions are immediate: Δ must be closed under isomorphisms and coproducts.

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One more natural condition suffices:

Theorem

Any component-based class Δ of finite relational structures or graphs, i.e. a class closed under

isomorphism,

finite coproducts, and

subcomponents/summands
(i.e. if $A + B$ is in Δ then so are A and B),

can be classified by a comonad.

Examples

The theorem applies to a wide variety of classes of structures studied in the literature:

- All classes of finite structures classified by our game comonads
- Many (many) examples of classes of structures for which a given graph parameter is bounded by a constant.
- Thus we obtain comonads for planar graphs, bipartite graphs, or graphs of max degree or clique-width bounded by a constant.

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Moreover, the comonad $\mathbb{C}$ we construct is *weakly initial* among the comonads classifying Δ, meaning that for any comonad $\mathbb{D}$ classifying Δ, there is a comonad morphism $\mathbb{C} \Rightarrow \mathbb{D}$.

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This initiality allows us to obtain a characterisation of comonads that classify *monotone nowhere dense classes*.

Classifying relations between structures

We say that a comonad $\mathbb{C}$ *classifies relation* $\asymp$ whenever $A \asymp B$ holds precisely whenever the cofree $\mathbb{C}$-coalgebras on A and B are isomorphic.

- For example, the comonad $\mathbb{E}_k$ classifies the relation that expresses that Duplicator has a winning strategy in the bijective k-round variant of the Ehrenfeucht–Fraïssé game and similarly $\mathbb{P}_k$ classifies the existence of a winning strategy in the bijective k-pebble game.
- Furthermore, it was recently shown that the relation classified by $\mathbb{E}_k$ admits a *Lovász-type theorem*.
- In particular, finite structures A, B have isomorphic $\mathbb{E}_k$-coalgebras iff they admit the same homomorphism counts from finite structures of tree-depth $\leq k$, i.e. when there is a bijection between $\hom(C, A)$ and $\hom(C, B)$ for every finite C of tree-depth $\leq k$.
- Similar Lovász-type theorems have been also shown for $\mathbb{P}_k$ and the pebble-relation comonads.

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In fact, we show that such comonad always has *finite rank*, which ensures that the category of coalgebras for the comonad is locally finitely presentable, and therefore, by a recent result of Luca Reggio [Reg21, Corollary 5.15], admits the following Lovász-type result:

Corollary

Let Δ be a component-based class of finite structures and let $\asymp$ be a binary relation on finite structures such that, for any two finite structures A, B,

$$A \asymp B \iff \hom(C, A) \cong \hom(C, B) \quad \textit{for all } C \in \Delta$$

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Examples:
The comonad which classifies planar graphs also classifies *quantum isomorphism*, the comonad for coproducts of cycles classifies *co-spectrality*, and the comonad for bipartite graphs classifies isomorphic bipartite double covers.

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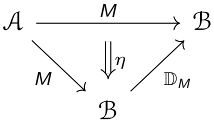
The *density comonad* of a functor $M\colon \mathcal{A} \to \mathcal{B}$ is a functor $\mathbb{D}_M\colon \mathcal{B} \to \mathcal{B}$ with a natural transformation $\eta\colon M \Rightarrow \mathbb{D}_M M$.



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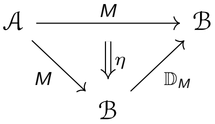

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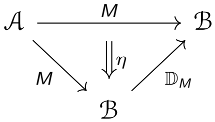

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Density comonads are special cases of left Kan extensions. They are not guaranteed to exist, but when $\mathcal{A}$ is small and $\mathcal{B}$ is cocomplete, they can be constructed by colimits.

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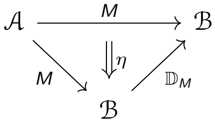

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We shall spell out the case of relational structures, including the comonad structure, to give a feel for the construction.

The construction on relational structures

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For a σ-structure B, the universe of $\mathbb{D}_M(B)$ consists of tuples

$$(A, f, x)$$

where $f\colon M(A) \to B$ is a homomorphism of relational structures and $x \in M(A)$. An n-ary relation R in σ is interpreted as the set of all tuples

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such that $R(x_1,\ \ldots,\ x_n)$ in $M(A)$.

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Given a homomorphism $g \colon B \to B'$, we have a homomorphism $\mathbb{D}_M(g) \colon \mathbb{D}_M(B) \to \mathbb{D}_M(B')$ defined by

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The counit $\epsilon_B\colon \mathbb{D}_M(B) \to B$ is the map

$$(M(A), h, x) \mapsto h(x)$$

and the comultiplication $\delta_B\colon \mathbb{D}_M(B) \to \mathbb{D}^2_M(B)$ is given by

$$(M(A), h, x) \mapsto (M(A), \lambda z \in M(A).(M(A), h, z), x)$$

The abstract classification theorem

An object C is *connected* iff $\hom(C, -)$ preserves coproducts.
We say that $\mathcal{B}$ is a *constituental category* if

> every object in $\mathcal{B}$ is (isomorphic to) a coproduct of connected objects, and
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We are given a functor $M : \mathcal{A} \to \mathcal{B}$ from a discrete category $\mathcal{A}$.
We say that an object of $\mathcal{B}$ is *essentially in* $\mathcal{A}$ if it is isomorphic to $M(A)$, for some A in $\mathcal{A}$.
We say that $\mathcal{A}$ is *component-based* if, whenever $B \in \mathcal{B}$ is essentially in $\mathcal{A}$ then so is every component of B.

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Theorem

Let $M \colon \mathcal{A} \to \mathcal{B}$ be a functor from a discrete component-based category $\mathcal{A}$ into a constituental category $\mathcal{B}$ such that the pointwise density comonad $\mathbb{D}_M$ exists. Then an object $b \in \mathcal{B}$ is isomorphic to a coproduct of objects essentially in $\mathcal{A}$ if and only if b admits a $\mathbb{D}_M$-coalgebra.

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Given a component-based class Δ of relational structures or graphs, let Δ_C be the subclass of Δ consisting of connected structures only. We then set $\mathcal{A}$ to be a discrete subcategory of $\mathcal{R}(\sigma)$ or Graph consisting of one representative from every isomorphism class in Δ_C. Since we picked only one representative from every equivalence class, the category $\mathcal{A}$ is small and, therefore, the density comonad $\mathbb{D}_M$, for the inclusion functor $M\colon \mathcal{A} \to \mathcal{R}(\sigma)$, exists because both $\mathcal{R}(\sigma)$ and Graph have all (small) coproducts. Observe that the comonad $\mathbb{D}_M$ classifies Δ.

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Indeed, by Theorem 3, a finite relational structure B has a $\mathbb{D}_M$-coalgebra if and only if there exist $C_1, \ldots, C_n$ in Δ_C such that $B \cong C_1 + \cdots + C_n$. In turn, this is equivalent to B being in Δ, which follows from being component-based as then $C_1 + \cdots + C_n$ is in Δ iff all the individual structures $C_1, \ldots, C_n$ are.

Examples

The category $\mathcal{R}(\sigma)$ of σ-structures is a constituental category. Therefore in applications we only need to check that a class Δ is closed under finite coproducts and summands. These are fairly weak conditions, satisfied by many well-known examples from the literature. In particular, this includes classes of finite structures closed under finite coproducts which are

1. monotone, i.e. closed under substructures,
2. hereditary, i.e. closed under taking induced substructures, or
3. closed under taking graph minors.

Further examples include

4. Fraïssé classes closed under free amalgamations, or
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On the other hand, any minor-closed class can be completed under finite coproducts. The resulting class is then still minor closed, and hence is classified by a comonad.

Graph parameters

A *graph parameter* is a mapping $\mu\colon \mathsf{Graph}_{\mathsf{fin}} \to \overline{\mathbb{R}}$, from finite graphs $\mathsf{Graph}_{\mathsf{fin}}$ to the extended reals $\overline{\mathbb{R}} = [-\infty, +\infty]$, which gives the same value to any two isomorphic graphs.
Moreover, we say that it is *standard* if $\mu(G_1 + G_2) = \max\{\mu(G_1), \mu(G_2)\}$.

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Lemma

Given a standard graph property $\mu\colon \mathsf{Graph}_{\mathsf{fin}} \to \{0, 1\}$*, the class of graphs* G *such that* $\mu(G) = 0$ *is closed under isomorphisms, finite coproducts, and summands. In fact, every such class is obtained from a standard graph property this way.* □

There is a comonad $\mathbb{C}^\mu$ which classifies the class Δ of finite graphs G such that $\mu(G) = 0$, for every standard graph property μ. We construct $\mathbb{C}^\mu$ explicitly, as the pointwise density comonad for the inclusion functor

$$\mathcal{A} \to \mathsf{Graph}, \tag{2}$$

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We use this to construct a sequence of comonads for a given standard graph parameter μ. For every extended real number k, we turn μ into a graph property

$$\mu_{\leq k} \colon \mathsf{Graph} \to \{0, 1\}$$

by setting $\mu_{\leq k}(G) = 0$ iff $\mu(G) \leq k$. Then, the density comonad $\mathbb{C}^\mu_k$, defined as $\mathbb{C}^\mu$ for $\mu := \mu_{\leq k}$, classifies finite graphs G such that $\mu(G) \leq k$.

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We use this to construct a sequence of comonads for a given standard graph parameter μ. For every extended real number k, we turn μ into a graph property

$$\mu_{\leq k} \colon \mathsf{Graph} \to \{0, 1\}$$

by setting $\mu_{\leq k}(G) = 0$ iff $\mu(G) \leq k$. Then, the density comonad $\mathbb{C}^\mu_k$, defined as $\mathbb{C}^\mu$ for $\mu := \mu_{\leq k}$, classifies finite graphs G such that $\mu(G) \leq k$.

Moreover, there is a linearly ordered chain of embeddings of discrete categories

$$\mathcal{A}_{-\infty} \hookrightarrow \cdots \hookrightarrow \mathcal{A}_k \hookrightarrow \mathcal{A}_l \hookrightarrow \cdots \hookrightarrow \mathcal{A}_{+\infty} \qquad \text{(with } k \leq l\text{)}$$

Hence, $(\mathbb{C}^\mu_k)_k$ is a graded comonad with the property that $\kappa^{\mathbb{C}^\mu}(G) = \mu(G)$ for every finite graph G.

Comparison with game comonads

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We show that discrete density comonads are *weakly initial* in the category of comonads that classify the same class.

Denote by $\mathbb{D}_\Delta$ the discrete density comonad constructed as above, for a component-based class Δ.

Proposition

Let Δ be a component-based class of relational structures or graphs and let $\mathbb{C}$ be a comonad that classifies a class Γ. Then, $\Delta \subseteq \Gamma$ if, and only if, there exists a comonad morphism $\mathbb{D}_\Delta \Rightarrow \mathbb{C}$.

Nowhere dense comonads

A class Δ is *somewhere dense* if there exists a natural number p such that, for every n, the p-th subdivision K_n^p of all edges in the clique graph K_n on n vertices is a subgraph of some graph in Δ.
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It is immediate that a monotone class of graphs Δ (i.e. a class closed under substructures) is somewhere dense if and only if $\mathrm{Cli}_p \subseteq \Delta$, for some p, where

$$\mathrm{Cli}_p = \{K_n^p \mid n \in \mathbb{N}\}$$

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Proposition

Assume $\mathbb{C}$ classifies a monotone class of graphs Δ. Then, Δ is nowhere dense if, and only if, there is no comonad morphism $\mathbb{D}_{\mathrm{Cli}_p} \Rightarrow \mathbb{C}$ for any $p \in \mathbb{N}$.

Lovász-type theorems for free

A classic result of Lovász [Lov67] says that two finite structures are isomorphic if and only if they admit the same number of homomorphisms from all finite structures.

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A classic result of Lovász [Lov67] says that two finite structures are isomorphic if and only if they admit the same number of homomorphisms from all finite structures.

In one type of generalisation, isomorphisms are replaced by an equivalence relation $\asymp$ on finite structures, and the class of all finite structures by a class of selected finite structures Δ.

Then a typical Lovász-type theorem expresses that, for finite structures A, B,

$$A \asymp B \iff \hom(C, A) \cong \hom(C, B) \quad \text{for every } C \in \Delta.$$

Δ	$\asymp$	reference
cycles	co-spectrality	(folklore)
trees	fractional isomorphism	[RSU94]
bipartite graphs	isomorphic bipartite double covers	[B1̈8, Lov12]
planar graphs	quantum isomorphism	[MR20]
tree-depth $\leq k$	Duplicator wins the bijective k-round Ehrenfeucht–Fraïssé game	[Gro20]
tree-width $< k$	Duplicator wins the bijective k-pebble game	[Dvo10]
path-width $< k$	Duplicator wins the bijective k-pebble relation game	[MS21]
admitting a k-pebble tree cover of height $\leq n$	Duplicator wins the bijective n-round k-pebble E.F. game	[DJR21]
synchronization trees of height $\leq k$	equivalence in graded modal logic of modal depth $\leq k$	[DJR21]
an inner-product compatible class	existence of a certain unitary map between homomorphism tensor spaces	[GRS21]

Density comonads yield Lovász theorems

For any component-based class Δ, we show that the discrete density comonad $\mathbb{C}$ that classifies Δ also classifies the relation $\asymp$, with $A \asymp B$ whenever $\hom(C, A) \cong \hom(C, B)$ for every $C \in \Delta$, thereby proving Corollary 2.

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The main ingredient of our proof is the following recent result due to Luca Reggio, proved abstractly for locally finitely presentable categories.

Theorem (Corollary 5.15 in [Reg21])

Let $\mathbb{C}$ be a comonad of finite rank on Graph *or $\mathcal{R}(\sigma)$. Then, for two finite structures A, B,*

$$F^{\mathbb{C}}(A) \cong F^{\mathbb{C}}(B) \quad \textit{iff} \quad \hom(C, A) \cong \hom(C, B),$$

for every finite C which admits a $\mathbb{C}$-coalgebra.

Finite rank theorem

Thus in order to prove Corollary 2, it is enough to show that the comonad $\mathbb{C}$ constructed in the proof of Theorem 1 has finite rank.

Theorem

Let $M: \mathcal{A} \to \mathcal{B}$ be a functor from a discrete category $\mathcal{A}$ to a constituental category $\mathcal{B}$ with finite coproducts and assume that the pointwise density comonad $\mathbb{D}_M$ for M exists. If all objects in the image of M are connected and finitely presentable then $\mathbb{D}_M$ has finite rank.

Final remarks

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For example, we can get new results by replacing one comonad by another.

Can we use the axiomatic framework for game comonads given by arboreal categories, and relate it to the density comonad construction to get similar transfer results?

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