Predicate and relation liftings for coalgebras with side effects

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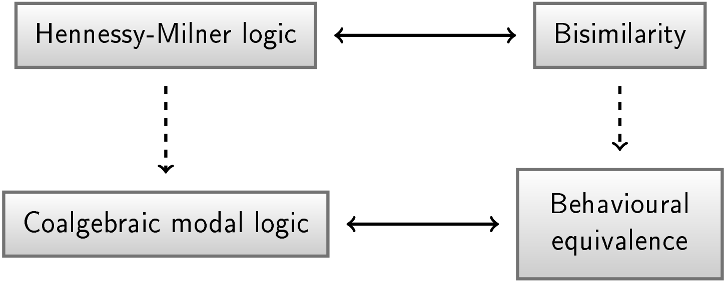
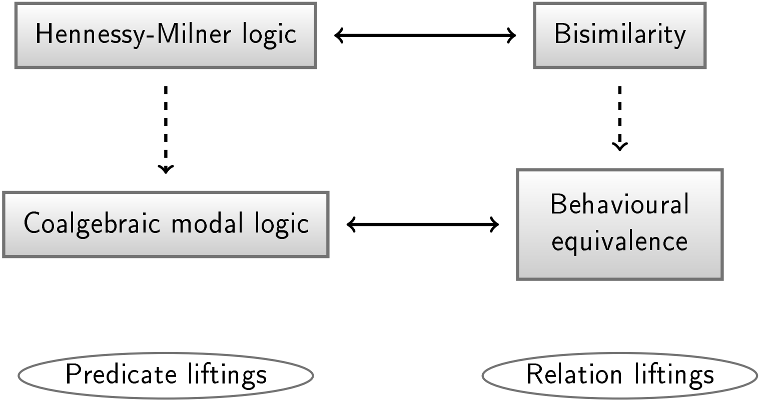
Motivation


Hennessy-Milner logic
Bisimilarity
Coalgebraic modal logic
Behavioural equivalence

Motivation



- Predicate and relations liftings for coalgebras living in (co)Kleisli categories and Eilenberg-Moore categories.

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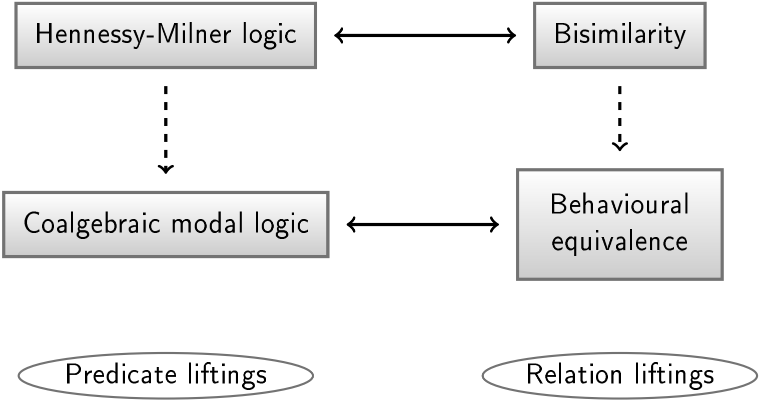
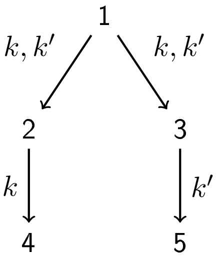


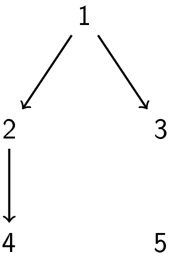
- Predicate and relations liftings for coalgebras living in (co)Kleisli categories and Eilenberg-Moore categories.
- Running example: *Conditional transition systems (CTSs)*.

Conditional transition systems (CTSs)

To model a software family

- CTSs are compact representation of a family of LTSs.
- Transitions are labelled with conditions $\mathbb{K}$ and action labels A. For simplicity, we let A to be singleton.





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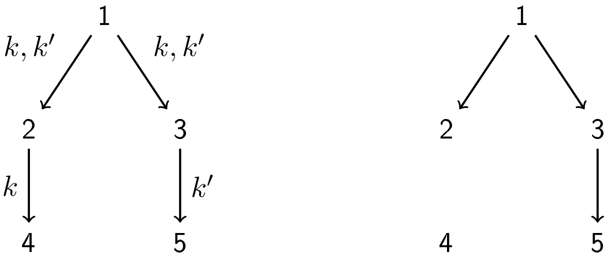

Coalgebraic modelling of CTSs

- $\mathbf{C} = \mathbf{coKl}(G)$ where $G = \mathbb{K} \times \_$:

$$\frac{X \in \mathbf{Set}}{X \in \mathbf{coKl}(G)} \qquad \frac{GX \xrightarrow{f} Y \in \mathbf{Set}}{X \xrightarrow{f} Y \in \mathbf{coKl}(G)}$$

- Branching type is given by $\mathbf{coKl}(G) \xrightarrow{\bar{\mathcal{P}}} \mathbf{coKl}(G)$, which maps $X \mapsto \mathcal{P}X$ and $GX \xrightarrow{f} Y \in \mathbf{Set}$:

$$G\mathcal{P}X \xrightarrow{\bar{\mathcal{P}}f} \mathcal{P}Y \qquad \bar{\mathcal{P}}f(k, U) = \{f(k, x) \mid x \in U\}.$$

- More abstractly, $\bar{\mathcal{P}}$ is a coKleisli extension of $\mathcal{P}$.
- A CTS is a coalgebra $X \xrightarrow{\alpha} \bar{\mathcal{P}}X \in \mathbf{coKl}(G)$.

Lifting of predicate liftings

- Systems are coalgebras living in $\mathbf{C}$, where $\mathbf{C}$ can be (co)Kleisli or Eilenberg-Moore category induced by some set (co)monad.

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "\bar{F}"] \arrow[d, "|_|"'] & \mathbf{C} \arrow[d, "|_|"] \\
\mathbf{Set} \arrow[r, "F"'] & \mathbf{Set}
\end{tikzcd}$$

- Predicates on sets are given by $\mathbf{Set}^{\mathsf{op}} \xrightarrow{\hat{\mathcal{P}}} \mathbf{Cat}$:

$$X \mapsto (\mathcal{P}X, \subseteq) \quad \text{and} \quad f \mapsto f^{-1}.$$

- So predicates on $\mathbf{C}$ is given by Φ defined as the composition

$$\mathbf{C}^{\mathsf{op}} \xrightarrow{|\_|} \mathbf{Set}^{\mathsf{op}} \xrightarrow{\hat{\mathcal{P}}} \mathbf{Cat}.$$

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Research questions

- Predicate liftings (Jacobs 2010) are indexed morphisms of type $\Phi \longrightarrow \Phi\bar{F}$.
- Can we lift indexed morphisms of type $\hat{\mathcal{P}} \longrightarrow \hat{\mathcal{P}}F$ to $\Phi \longrightarrow \Phi\bar{F}$?

Lifting of predicate liftings

When $\mathbf{C} = \mathbf{coKl}(G)$ or $\mathbf{C} = \mathbf{EM}(T)$

- Let $\bar{F}$ be a coKleisli extension/Eilenberg-Moore lifting of F.

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "\bar{F}"] \arrow[d, "|_|"'] \arrow[dr, phantom, "\Downarrow \gamma"] & \mathbf{C} \arrow[d, "|_|"] \\
\mathbf{Set} \arrow[r, "F"'] & \mathbf{Set}
\end{tikzcd}$$

- Suppose $\hat{\mathcal{P}} \xrightarrow{\sigma} \hat{\mathcal{P}}F$ is given.

Then we can define $\Phi \xrightarrow{\lambda} \Phi\bar{F}$ as follows:

$$\Phi X = \hat{\mathcal{P}}|X| \xrightarrow{\sigma_{|X|}} \hat{\mathcal{P}}F|X| \xrightarrow{\gamma_X^*} \hat{\mathcal{P}}|FX| = \Phi\bar{F}X.$$

Lifting of predicate liftings

When $\mathbf{C} = \mathbf{Kl}(T)$

- Let $\bar{F}$ be a Kleisli extension of F.

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "\bar{F}"] \arrow[d, "|_|"'] \arrow[dr, phantom, "\Uparrow"] & \mathbf{C} \arrow[d, "|_|"] \\
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\end{tikzcd}$$

As an example, G is typically associated with the branching type of deterministic version of the system of interest. I.e., in the context of NDA, $G = \_^{A} \times 2$ when $F = A \times \_ + 1$.

Box modality for CTSs

$$\begin{array}{ccc} \mathbf{coKl}(G) & \xrightarrow{\bar{\mathcal{P}}} & \mathbf{coKl}(G) \\ {\scriptstyle |\_|}\downarrow & & \downarrow{\scriptstyle |\_|} \\ \mathbf{Set} & \xrightarrow[\mathcal{P}]{} & \mathbf{Set} \end{array}$$

- We need first a predicate lifting for $F = \mathcal{P}$ on $\mathbf{Set}$. To this end, take $\hat{\mathcal{P}} \xrightarrow{\sigma} \hat{\mathcal{P}}F$ for box modality.
- Our γ is given by the distributive law:

$$GFX = \mathbb{K} \times \mathcal{P}X \xrightarrow{\gamma_X} \mathcal{P}(\mathbb{K} \times X) = FGX$$

  that maps a pair $(k, U) \mapsto \{k\} \times U$.
- Now upon computing λ we find that:

$$|\alpha|^{-1}\lambda_X\bar{U} = \{(k,x) \mid \forall_{x'}\ x \xrightarrow{k} x' \implies (k,x') \in \bar{U}\},$$

  where $\bar{U} \subseteq \mathbb{K} \times X$ or $\bar{U} \in \Phi X$.

Relation liftings

Under some restrictions, relation liftings can be lifted in the same manner as predicate liftings.

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Working assumption

- Our working category $\mathbf{C}$ has product $\otimes$. Thus we can define an indexed category Ψ of relations as the composition:

$$\mathbf{C}^{\mathsf{op}} \xrightarrow{\_\otimes\_} \mathbf{C}^{\mathsf{op}} \xrightarrow{|\_|} \mathbf{Set}^{\mathsf{op}} \xrightarrow{\hat{\mathcal{P}}} \mathbf{Cat}.$$

- The elements of ΨX are called 'abstract' relations.

Relation liftings

Under some restrictions, relation liftings can be lifted in the same manner as predicate liftings.

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\mathbf{C} \arrow[r, "\bar{F}"] \arrow[d, "|_|"'] & \mathbf{C} \arrow[d, "|_|"] \\
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Research question

- When can we lift a relation lifting $\hat{\mathcal{P}}(\_ \times \_) \longrightarrow \hat{\mathcal{P}}G(\_ \times \_)$ to an indexed morphism $\Psi \longrightarrow \Psi\bar{F}$?

Prerequisite on lifting a relation lifting

Recall that $\hat{\mathcal{P}}$ is a bifibration

So for any $X \xrightarrow{f} Y \in \mathbf{Set}$ we have

$$\begin{tikzcd} \hat{\mathcal{P}}X \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & \hat{\mathcal{P}}Y. \arrow[l, shift left, "f^{-1}"] \end{tikzcd}$$

Recall that $\mathbf{C}$ has product $\otimes$

Moreover, projection functions $X \otimes X \xrightarrow{\pi_1^X, \pi_2^X} X \in \mathbf{C}$ induce:

$$|X \otimes X| \xrightarrow{\langle|\pi_1^X|, |\pi_2^X|\rangle} |X| \times |X| \in \mathbf{Set}.$$

Therefore, for any $X \in \mathbf{C}$, we have

$$\begin{tikzcd} \Psi X = \hat{\mathcal{P}}|X \otimes X| \arrow[r, shift left, "\langle|\pi_1^X|, |\pi_2^X|\rangle_!"] \arrow[r, phantom, "\perp"] & \hat{\mathcal{P}}(|X| \times |X|). \arrow[l, shift left, "\langle|\pi_1^X|, |\pi_2^X|\rangle^{-1}"] \end{tikzcd}$$

Our recipe of lifting a relation lifting

Given a diagram

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\bar{F}"] \arrow[d, "|\_|"'] \arrow[dr, phantom, "\Downarrow\gamma"] & \mathbf{C} \arrow[d, "|\_|"] \\ \mathbf{Set} \arrow[r, "G"'] & \mathbf{Set} \end{tikzcd}$$

then we can lift an indexed morphism $\hat{\mathcal{P}}(\_ \times \_) \xrightarrow{\sigma} \hat{\mathcal{P}}G(\_ \times \_)$:

$$\begin{tikzcd} \hat{\mathcal{P}}(|X| \times |X|) \arrow[r, "\sigma_{|X|}"] & \hat{\mathcal{P}}(G|X| \times G|X|) \arrow[r, "(\gamma_X \times \gamma_X)^{-1}"] & \hat{\mathcal{P}}(|\bar{F}X| \times |\bar{F}X|) \arrow[d, "\langle \pi_1^{FX}, \pi_2^{FX} \rangle^{-1}"] \\ \Psi X \arrow[u, "\langle \pi_1^X, \pi_2^X \rangle_!"] \arrow[rr] & & \Psi \bar{F} X. \end{tikzcd}$$

Lifting of relation liftings

Result

If the following commutative square

$$\begin{tikzcd}
{|X \otimes X|} \arrow[r, "\langle |\pi_1^X|, |\pi_2^X| \rangle"] \arrow[d, "|f \otimes f|"'] & {|X| \times |X|} \arrow[d, "|f| \times |f|"] \\
{|Y \otimes Y|} \arrow[r, "\langle |\pi_1^Y|, |\pi_2^Y| \rangle"] & {|Y| \times |Y|}
\end{tikzcd}$$

is a weak pullback in $\mathbf{Set}$ (for every $X \xrightarrow{f} Y \in \mathbf{C}$) then our recipe indeed defines an indexed morphism.

Corollary

When $\mathbf{C}$ is either a Kleisli or an Eilenberg-Moore category, then the above property is always satisfied.

Behavioural equivalence fibrationally

Given $X \xrightarrow{\alpha} \bar{F}X \in \mathbf{C}$ and $\mathbf{C}^{\mathsf{op}} \xrightarrow{\Psi} \mathbf{Cat}$:

$$\begin{tikzcd} \mathbf{Coalg}(\bar{F}) \arrow[r] & \mathbf{C} \arrow[r, "\bar{F}"'] & \mathbf{C} \end{tikzcd}$$

Behavioural equivalence fibrationally

Given $X \xrightarrow{\alpha} \bar{F}X \in \mathbf{C}$ and $\mathbf{C}^{\mathsf{op}} \xrightarrow{\Psi} \mathbf{Cat}$:

$$\begin{tikzcd}
 & \int \Psi \arrow[d, "p"] & \\
\mathbf{Coalg}(\bar{F}) \arrow[r] & \mathbf{C} \arrow[r, "\bar{F}"'] & \mathbf{C}
\end{tikzcd}$$

Ψ gives rise to a fibration $\int \Psi \xrightarrow{p} \mathbf{C}$

$$\frac{X \in \mathbf{C} \wedge R \in \Psi X}{(X, R) \in \int \Psi} \qquad \frac{X \xrightarrow{f} Y \in \mathbf{C} \wedge R \subseteq f^* S \in \Psi X}{(X, R) \xrightarrow{f} (Y, S) \in \int \Psi}$$

Behavioural equivalence fibrationally

Given $X \xrightarrow{\alpha} \bar{F}X \in \mathbf{C}$ and $\mathbf{C}^{\mathsf{op}} \xrightarrow{\Psi} \mathbf{Cat}$:

$$\begin{tikzcd}
 & \int \Psi \arrow[r, "\bar{F}_\lambda"] \arrow[d, "p"] & \int \Psi \arrow[d, "p"] \\
\mathbf{Coalg}(\bar{F}) \arrow[r] & \mathbf{C} \arrow[r, "\bar{F}"'] & \mathbf{C}
\end{tikzcd}$$

Indexed morphism gives rise to a map of fibrations

An indexed morphism $\Psi \xrightarrow{\lambda} \Psi\bar{F}$ induces $\bar{F}_\lambda$:

$$\bar{F}_\lambda(X, R) = (\bar{F}X, \lambda_X R) \qquad \bar{F}_\lambda f = \bar{F}f$$

Behavioural equivalence fibrationally

Given $X \xrightarrow{\alpha} \bar{F}X \in \mathbf{C}$ and $\mathbf{C}^{\mathsf{op}} \xrightarrow{\Psi} \mathbf{Cat}$:

$$
\begin{tikzcd}
\mathbf{Coalg}(\bar{F}_\lambda) = \int \Psi_\lambda^{\bar{F}} \arrow[r] \arrow[d, "p_\lambda^{\bar{F}}"'] & \int \Psi \arrow[r, "\bar{F}_\lambda"] \arrow[d, "p"] & \int \Psi \arrow[d, "p"] \\
\mathbf{Coalg}(\bar{F}) \arrow[r] & \mathbf{C} \arrow[r, "\bar{F}"'] & \mathbf{C}
\end{tikzcd}
$$

$\mathbf{Coalg}(\bar{F}_\lambda)$ is again a fibration (Jacobs 2010)

Moreover, there is an indexed category $\mathbf{Coalg}(\bar{F})^{\mathsf{op}} \xrightarrow{\Psi^{\bar{F}_\lambda}} \mathbf{Cat}$ that $(X, \alpha) \mapsto \mathbf{Coalg}(\alpha^* \circ \lambda_X)$. In the context of a CTS $X \xrightarrow{\alpha} \bar{\mathcal{P}}X$,

R is a conditional bisimulation $\iff$ $R \subseteq \alpha^* \lambda_X R$.

Behavioural equivalence fibrationally

Given $X \xrightarrow{\alpha} \bar{F}X \in \mathbf{C}$ and $\mathbf{C}^{\mathsf{op}} \xrightarrow{\Psi} \mathbf{Cat}$:

$$
\begin{tikzcd}
\mathbf{Coalg}(\bar{F}_\lambda) = \int \Psi^{\bar{F}}_\lambda \arrow[r] \arrow[d, shift right=2ex, "p^{\bar{F}}_\lambda"'] \arrow[d, phantom, "\dashv"] & \int \Psi \arrow[r, "\bar{F}_\lambda"] \arrow[d, "p"] & \int \Psi \arrow[d, "p"] \\
\mathbf{Coalg}(\bar{F}) \arrow[r] \arrow[u, shift right=2ex, "\mathbb{1}^\lambda"'] & \mathbf{C} \arrow[r, "\bar{F}"'] & \mathbf{C}
\end{tikzcd}
$$

The behavioural conformance functor $\mathbb{1}^\lambda$

- It exists when $\Psi^{\bar{F}_\lambda}$ has indexed final objects.
- E.g., in the case of CTS, $\mathbb{1}^\lambda(X, \alpha) = (X, \alpha, \leftrightarrows_X)$.

Conclusions

Objects of our study are *coalgebras with side effects*, i.e., those living in (co)Kleisli/Eilenberg-Moore categories.

- Recipe to construct predicate/relation liftings.
- Conditions when behavioural equivalence can be characterised as coalgebras living in the fibre of 'abstract' relations.
- Extended the dual adjunction framework for fibrations by (Kupke and Rot 2020) to coalgebras with side effects.

Future work

- Coalgebraic games (Mika-Michalski 2022) / *Chase's talk on behavioural equivalence games*.
- Shift from behavioural equivalences to distances.

B. P. F. Jacobs. *Predicate Logic for Functors and Monads.* Available from author's website. 2010. url: http://www.cs.ru.nl/~bart/PAPERS/predlift-indcat.pdf.

C. Kupke and J. Rot. 'Expressive Logics for Coinductive Predicates'. In: *28th EACSL Annual Conference on Computer Science Logic (CSL 2020)*. Ed. by M. Fernández and A. Muscholl. Vol. 152. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2020, 26:1–26:18.

Christina Mika-Michalski. 'System Verification Via Generic Games: Behavioural Equivalence and Model Checking Games'. PhD thesis. Feb. 2022.

Extension semantics

An extension semantics γ that connects the KL-law $FT \xrightarrow{\vartheta} TF$ and the EM-law $TG \xrightarrow{\vartheta'} GT$ is a natural transformation $TF \xrightarrow{\gamma} GT$ satisfying the following two commutative diagrams:

$$
\begin{tikzcd}
TFTX \arrow[r, "\gamma_{TX}"'] \arrow[d, "T\vartheta_X"'] & GTTX \arrow[dd, "G\mu_X"] \\
TTFX \arrow[d, "\mu_{FX}"'] & \\
TFX \arrow[r, "\gamma_X"] & GTX
\end{tikzcd}
$$

$$
\begin{tikzcd}
TTFX \arrow[r, "T\gamma_X"'] \arrow[dd, "\mu_{FX}"'] & TGTX \arrow[d, "\vartheta'_X"] \\
& GTTX \arrow[d, "G\mu_X"] \\
TFX \arrow[r, "\gamma_X"] & GTX
\end{tikzcd}
$$

Kupke-Rot setup

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\int \Psi \arrow[r, shift left, "\bar{\mathcal{S}}"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "p"] & \mathbf{A}^{\mathsf{op}} \arrow[l, shift left, "\mathrm{Eq} \circ \mathcal{T}"] \arrow[d, equal] \\
\mathbf{C} \arrow[r, shift left, "\mathcal{S}"] \arrow[r, phantom, "\perp"] \arrow[u, shift left, "\mathrm{Eq}"] & \mathbf{A}^{\mathsf{op}} \arrow[l, shift left, "\mathcal{T}"]
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=huge, row sep=huge]
\int \Psi \arrow[r, shift left, "\mathcal{S} \circ \mathbb{Q}"] \arrow[r, phantom, "\perp"] \arrow[d, shift right=3, "\mathbb{Q}"'] \arrow[d, shift left=3, "p"] & \mathbf{A}^{\mathsf{op}} \arrow[l, shift left, "\mathrm{Eq} \circ \mathcal{T}"] \arrow[d, equal] \\
\mathbf{C} \arrow[r, shift left, "\mathcal{S}"] \arrow[r, phantom, "\perp"] \arrow[u, "\mathrm{Eq}"] \arrow[u, phantom, shift right=2, "\dashv"] & \mathbf{A}^{\mathsf{op}} \arrow[l, shift left, "\mathcal{T}"]
\end{tikzcd}
$$

Result

If $\bar{\mathcal{T}} = \mathrm{Eq} \circ \mathcal{T}$ has a left adjoint $\bar{\mathcal{S}}$, the logic (L, δ) is adequate. Moreover it is expressive if $|\delta_{\mathcal{A}}|$ is injective.