Graded Monads and Behavioural Equivalence Games

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- Graded semantics: universal coalgebra + graded monads
  (Milius/Pattinson/Schröder, CALCO 2015)

- Equivalence games in the spirit of the bisimilarity game on LTS:

Position	Player	Admissible Moves
$(x, y) \in X^2$	D	$\{Z \subseteq X^2 \mid Z \text{ a local bisimulation at } (x, y)\}$
$Z \subseteq X^2$	S	$Z = \{(x, y) \in X^2 \mid (x, y) \in Z\}$

Main theorem

$$x \sim_{\mathcal{S}} y \iff \text{D wins the } n\text{-round } \mathcal{S}\text{-game for all } n \in \omega$$

Motto: graded monads offer a uniform view on behavioural equivalence games

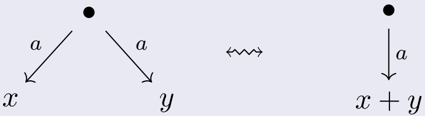
Graded theories and graded semantics

- graded theories $\leftrightsquigarrow$ (finitary) graded monads on Set:
  - ▷ *graded signature*: algebraic signature + *depth* on operations
  - ▷ *uniform-depth equations*: pairs of terms of the same depth
- Graded semantics on coalgebras for $G\colon \mathsf{Set} \to \mathsf{Set}$:
  - ▷ $\alpha\colon G \to M_1$ (= depth-1 terms modulo derivable equality)
  - ▷ e.g. bisimilarity, (ready) similarity, (probabilistic) trace equivalence

Graded theory of $\mathcal{A}$-traces $\rightsquigarrow$ trace equivalence on $\mathcal{A}$-LTS

- Depth-0: operations/equations of join semilattices
- Depth-1: unary *actions* $a(-)$ satisfying $\boxed{a(x) + a(y) = a(x + y)}$

$\boxed{a(x) + a(y)}$



$\boxed{a(x + y)}$

$\mathcal{G}_n$ captures *precisely* depth-n $\mathcal{S}$-equivalence on

$$\gamma \colon X \to GX \qquad \rightsquigarrow \qquad \bar{\gamma} = (X \xrightarrow{\gamma} GX \xrightarrow{\alpha} M_1 X)$$

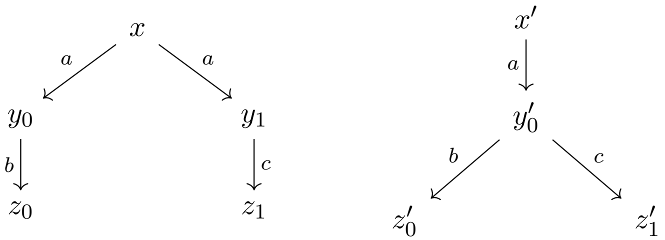
...starting from $\eta(x) = \eta(y)$ for target states x, y:

Position	Player	Admissible Moves
$(s,t) \in (M_0X)^2$	D	$\{Z \subseteq X^2 \mid Z \vdash_1 s\bar{\gamma} = t\bar{\gamma}\}$
$Z \subseteq (M_0X)^2$	S	$Z = \{(s,t) \in (M_0X)^2 \mid (s,t) \in Z\}$

$$\text{Play of } \mathcal{G}_n : \quad (s,t)\ Z_1\ (s_1,t_1)\ \dots\ Z_n\ \boxed{(s_n,t_n)}$$

Slogan: equivalence games play out equational proofs in graded theories

$x \sim_{\text{trace}} x'$? Play the 2-round trace game!



- At (x, x'), D plays $Z_1 := \{y_0 + y_1 = y'_0\}$

  admissible: $Z_1 \vdash_1 a(y_0) + a(y_1) = a(y'_0)$?
- At Z_1, S plays $(y_0 + y_1 = y'_0) \in Z_1$
- At $(y_0 + y_1, y)$, D plays $Z_2 := \{z_0 = z'_0, z_1 = z'_1\}$

  admissibile: $Z_2 \vdash_1 b(z_0) + c(z_1) = b(z'_0) + c(z'_1)$?
- Again, S plays a challenge from Z_2 inducing $\boxed{(x, x')\ Z_1\ (y_0 + y_1, y'_0)\ Z_2\ (z_i, z'_i)}$

  D wins because $* = *$ is valid in JSL

Concluding remarks

Also available in the paper on arXiv:

- Pre-determinization under graded semantics
  e.g. determinization of *serial* LTS under simulation equivalence
- notion of infinite-depth equivalence for graded semantics
  e.g. infinite-depth (ready) trace equivalence on *serial* LTS

Future Work

- algorithmic/complexity matters of graded semantics
- extensions beyond equivalences (e.g. preorders, distances, etc.)

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