Graded Monads and Behavioural Equivalence Games

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• **Graded semantics**: universal coalgebra + **graded monads**
  (Milius/Pattinson/Schröder, CALCO 2015)

• **Equivalence games** in the spirit of the bisimilarity game on LTS:

<table>
<thead>
<tr>
<th>Position</th>
<th>Player</th>
<th>Admissible Moves</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x, y) \in X^2)</td>
<td>D</td>
<td>({ Z \subseteq X^2 \mid Z \text{ a local bisimulation at } (x, y) })</td>
</tr>
<tr>
<td>(Z \subseteq X^2)</td>
<td>S</td>
<td>(Z = {(x, y) \in X^2 \mid (x, y) \in Z})</td>
</tr>
</tbody>
</table>

**Main theorem**

\[ x \sim_S y \iff \text{D wins the } n\text{-round } S\text{-game for all } n \in \omega \]

**Motto**: graded monads offer a uniform view on behavioural equivalence games
Graded theories and graded semantics

- Graded theories \(\leftrightarrow\) (finitary) graded monads on \(\text{Set}\):
  - graded signature: algebraic signature + depth on operations
  - uniform-depth equations: pairs of terms of the same depth

- Graded semantics on coalgebras for \(G: \text{Set} \to \text{Set}\):
  - \(\alpha: G \to M_1\) (= depth-1 terms modulo derivable equality)
  - e.g. bisimilarity, (ready) similarity, (probabilistic) trace equivalence

Graded theory of \(A\)-traces \(\leadsto\) trace equivalence on \(A\)-LTS

- Depth-0: operations/equations of join semilattices
- Depth-1: unary actions \(a(\cdot)\) satisfying \(a(x) + a(y) = a(x + y)\)
Overview of the generic $n$-round game $\mathcal{G}_n$

$\mathcal{G}_n$ captures precisely depth-n $S$-equivalence on

$$\gamma: X \rightarrow GX \rightsquigarrow \tilde{\gamma} = (X \xrightarrow{\gamma} GX \xrightarrow{\alpha} M_1 X)$$

...starting from $\eta(x) = \eta(y)$ for target states $x, y$:

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<td>$(s, t) \in (M_0X)^2$</td>
<td>D</td>
<td>${ Z \subseteq X^2 \mid Z \vdash_1 s\tilde{\gamma} = t\tilde{\gamma} }$</td>
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<tr>
<td>$Z \subseteq (M_0X)^2$</td>
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Play of $\mathcal{G}_n$: $(s, t) \; Z_1 \; (s_1, t_1) \; \ldots \; Z_n \; (s_n, t_n)$

**Slogan:** equivalence games play out equational proofs in graded theories
$x \sim_{\text{trace}} x'$? Play the 2-round trace game!

- At $(x, x')$, D plays $Z_1 := \{y_0 + y_1 = y'_0\}$
  
admissibile: $Z_1 \vdash_1 a(y_0) + a(y_1) = a(y'_0)$?

- At $Z_1$, S plays $(y_0 + y_1 = y'_0) \in Z_1$

- At $(y_0 + y_1, y)$, D plays $Z_2 := \{z_0 = z'_0, z_1 = z'_1\}$
  
admissibile: $Z_2 \vdash_1 b(z_0) + c(z_1) = b(z'_0) + c(z'_1)$?

- Again, S plays a challenge from $Z_2$ inducing $[(x, x') \ Z_1 (y_0 + y_1, y'_0) \ Z_2 (z_i, z'_i)]$

  D wins because $* = *$ is valid in JSL
Concluding remarks

Also available in the paper on arXiv:

- **Pre-determinization** under graded semantics
  e.g. determinization of *serial* LTS under simulation equivalence
- **notion of infinite-depth equivalence** for graded semantics
  e.g. infinite-depth (ready) trace equivalence on *serial* LTS

**Future Work**

- algorithmic/complexity matters of graded semantics
- extensions beyond equivalences (e.g. preorders, distances, etc.)

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