Coalgebraic Semantics for Nominal Automata

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Overview

An exercise in coalgebra:

1. Kleisli style coalgebraic trace semantics
   á la Jacobs, Hasuo, and Sokolova revisited
2. The language semantics of nominal automata arises as an instance of
   - Kleisli-style coalgebraic trace semantics
   - Eilenberg-Moore-style coalgebraic language semantics (generalized determinization)

   The coincidence is an instance of Jacobs, Silva, and Sokolova's extension semantics.

Disclaimer

No surprising results for the cognoscenti.

But some pitfalls and surprises in the proofs.

Part 1:
Coalgebraic Trace Semantics

The Motivating Example

- Labelled transition systems (with explicit termination) are coalgebras
$$X \to \mathcal{P}(1 + \Sigma \times X) = TFX$$
- the initial algebra $\mu F = \big(1 + \Sigma \times \Sigma^* \xrightarrow{[\mathsf{nil},\mathsf{cons}]} \Sigma^*\big)$ yields a terminal coalgebra in $\mathsf{Kl}(\mathcal{P})$
- the unique Kleisli-morphism $X \to \mathcal{P}(\Sigma^*)$ is $x \mapsto \{w \in \Sigma^* : x \xrightarrow{w} y{\downarrow}\}$

Reasons:

- F has an extension $\bar{F}\colon \mathsf{Kl}(T) \to \mathsf{Kl}(T)$
- $\mathsf{Kl}(\mathcal{P})$ is enriched in complete partial orders
- $\bar{F}$ is finitary and locally monotone/continuous
- A Smyth-Plotkin-type limit-colimit-coincidence argument shows that the initial-algebra and terminal coalgebra (chains) coincide.

DCPO Enriched Categories

Definition

1. A category $\mathcal{C}$ is left-strictly $\mathsf{DCPO}_\perp$-enriched provided that each hom-set is a dcpo with bottom, and composition is monotone, preserves directed joins and left-strict: for every morphism f and appropriate directed sets of morphisms g_i $(i \in D)$ we have

$$\perp \cdot f = \perp, \quad f \cdot \bigvee_{i \in D} g_i = \bigvee_{i \in D} f \cdot g_i, \quad \left(\bigvee_{i \in D} g_i\right) \cdot f = \bigvee_{i \in D} g_i \cdot f.$$

2. A functor on $\mathcal{C}$ is locally monotone if its restrictions $\mathcal{C}(A, B) \to \mathcal{C}(FA, FB)$ to the hom-sets are monotone.

Example

- $\mathsf{Kl}(\mathcal{P})$ is left-strictly $\mathsf{DCPO}_\perp$-enriched
- $\bar{F}X = 1 + \Sigma \times X$ is locally monotone

Canonical Fixed Points

Theorem

Let $\mathcal{C}$ be left-strictly $\mathsf{DCPO}_\perp$-enriched and $F\colon \mathcal{C} \to \mathcal{C}$ locally monotone.

Then: $FI \xrightarrow{\iota} I$ initial algebra $\implies$ $I \xrightarrow{\iota^{-1}} FI$ terminal coalgebra.

Freyd proved this for locally continuous functors using Kleene's theorem.

Proof. **Existence:** given $C \xrightarrow{\gamma} FC$, take

$$\mathcal{C}(C, I) \xrightarrow{\Phi} \mathcal{C}(C, I) \qquad \left(C \xrightarrow{h} I\right) \longmapsto \left(C \xrightarrow{\gamma} FC \xrightarrow{Fh} FI \xrightarrow{\iota} I\right).$$

Since $\mathcal{C}(C, I)$ is a DCPO with $\perp$ and g is monotone, it has a least fixed point $\mu\Phi = h$.

This is a coalgebra homomorphism:

$$h = \iota \cdot Fh \cdot \gamma \qquad \iff \qquad \begin{array}{ccc} C & \xrightarrow{h} & I \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\iota^{-1}} \\ FC & \xrightarrow{Fh} & FI \end{array}$$

Canonical Fixed Points

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Then: $FI \xrightarrow{\iota} I$ initial algebra $\implies$ $I \xrightarrow{\iota^{-1}} FI$ terminal coalgebra.

Proof. **Unicity:** given $h'\colon (C,\gamma) \to (\mu F, \iota^{-1})$ form

$$\begin{array}{ccc} \mathcal{C}(I,I) & \xrightarrow{\Psi} & \mathcal{C}(I,I) \\ {\scriptstyle \mathcal{C}(h',I)}\downarrow & & \downarrow{\scriptstyle \mathcal{C}(h',I)} \\ \mathcal{C}(C,I) & \xrightarrow{\Phi} & \mathcal{C}(C,I) \end{array}$$

$$\begin{aligned} (I \xrightarrow{k} I) &\longmapsto (I \xrightarrow{\iota^{-1}} FI \xrightarrow{Fk} FI \xrightarrow{\iota} I) \\ (I \xrightarrow{k} I) &\longmapsto (C \xrightarrow{h'} I \xrightarrow{k} I) \\ (C \xrightarrow{h} I) &\longmapsto (C \xrightarrow{\gamma} FC \xrightarrow{Fh} FI \xrightarrow{\iota} I) \end{aligned}$$

- $\mathcal{C}(h', I)$ is strict and monotone, whence preserves least fixed points:
  $$\mathcal{C}(h',I)(\mu\Psi) = \mu\Phi \qquad \text{(uniformity)}$$
- $\mu\Psi = \mathrm{id}_I$ unique fixed point since $F(\mu F) \xrightarrow{\iota} \mu F$ initial algebra
- Thus: $\mu\Phi = \mathcal{C}(h',I)(\mathrm{id}_I) = \mathrm{id}_I \cdot h' = h'$. □

Instance: Kleisli Categories

Theorem (Hermida and Jacobs)

Let $T\colon \mathcal{C} \to \mathcal{C}$ be a monad and let $F\colon \mathcal{C} \to \mathcal{C}$ have an extension $\bar{F}\colon \mathsf{Kl}(T) \to \mathsf{Kl}(T)$. Then

$J\colon \mathcal{C} \to \mathsf{Kl}(T)$

$$FI \xrightarrow{\iota} I \text{ initial algebra} \iff J\iota = \left(FI \xrightarrow{\iota} I \xrightarrow{\eta_I} TI\right) \text{ initial } \bar{F}\text{-algebra.}$$

Theorem

Let T be a monad and F a functor on $\mathcal{C}$. Assume that

- $\mathsf{Kl}(T)$ is left-strictly $\mathsf{DCPO}_\perp$-enriched,
- F has a locally monotone extension $\bar{F}\colon \mathsf{Kl}(T) \to \mathsf{Kl}(T)$.

Then (I, ι) initial F-algebra $\implies$ $(I, J\iota^{-1})$ terminal $\bar{F}$-coalgebra.

Proof. Immediate from the previous two theorems. □

Coalgebraic Trace Semantics

Definition

Given T and F as before and a coalgebra $X \xrightarrow{c} TFX$.

The coalgebraic trace map is given by final semantics in $\mathsf{Kl}(T)$:

$$\begin{array}{ccc} X & \xrightarrow{\mathsf{tr}_c} & \mu F \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle J\iota^{-1}} \\ \bar{F}X & \xrightarrow{\bar{F}\mathsf{tr}_c} & \bar{F}(\mu F) \end{array}$$

Example: Traces of labelled transition systems

Given $X \xrightarrow{c} \mathcal{P}(1 + \Sigma \times X)$, we have $X \xrightarrow{\mathsf{tr}_c} \mathcal{P}(\Sigma^*)$ given by

$$\mathsf{tr}_c(x) = \{w \in \Sigma^* : x \xrightarrow{w} y{\downarrow}\} \qquad \text{for all } x \in X.$$

Part 2:
Nominal Automata and Their Language Semantics

Nominal Sets (aka Sets with Atoms)

- $\mathbb{A}$ = infinite set of atoms or names ← modelling data or resources
- $\mathrm{Perm}(\mathbb{A}) = \{\pi \mid \pi \colon \mathbb{A} \to \mathbb{A}$ finite permutation$\}$
- Nominal set = set X with $\mathrm{Perm}(\mathbb{A})$-action $(\pi, x) \mapsto \pi \cdot x$ such that every $x \in X$ has a finite support S ← if $\forall s \in S.\, \pi(s) = s$, then $\pi \cdot x = x$
- $\mathrm{supp}(X)$ = least support of X
- **Nom** category of nominal set and equivariant maps $f \colon X \to Y$:

$$f(\pi \cdot x) = \pi \cdot f(x) \qquad \text{for every } \pi \in \mathrm{Perm}(\mathbb{A}),\ x \in X.$$

Functors on Nom

- Coproducts and finite products as in sets: $X \times Y$, $X + Y$
- $\mathcal{P}_{\mathsf{fs}} X = \{Y \subseteq X \mid Y \text{ finitely supported}\}$, $\quad \pi \cdot Y = \{\pi \cdot y \mid y \in Y\}$
- $\mathcal{P}_{\mathsf{ufs}} X = \{Y \subseteq X \mid Y$ uniformly finitely supported$\}$

  $\mathsf{supp}(Y) = \bigcup_{y \in Y} \mathsf{supp}(y)$

- Abstraction functor: $[\mathbb{A}]X = \mathbb{A} \times X/\!\sim$ where

$c \notin \{a, b\} \cup \mathsf{supp}(x) \cup \mathsf{supp}(y)$

$$(a, x) \sim (b, y) \quad :\Longleftrightarrow \quad (a\,c) \cdot x = (b\,c) \cdot y \text{ for some fresh } c$$

Notation: $\langle a \rangle x = [(a, x)]_\sim$. For an equivariant $f\colon X \to Y$

$$[\mathbb{A}]f\colon [\mathbb{A}]X \to [\mathbb{A}]Y \qquad \langle a \rangle x \longmapsto \langle a \rangle f(x).$$

Intuition: $[\mathbb{A}]X \to X$ is a name-binding operation

e.g. λ-abstraction

Nominal Automata (which) are Coalgebras

NOFAs (Bojańczyk, Klin, Lasota 2014)

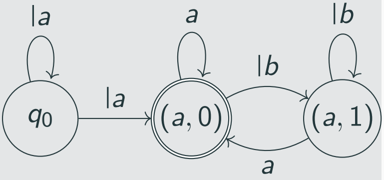
- Non-deterministic orbit-finite automata (NOFAs) are coalgebras $Q \to \{0,1\} \times \mathcal{P}_{\mathsf{fs}}(\mathbb{A} \times Q)$.
- Every state $q \in Q$ accepts its data language $L_q \subseteq \mathbb{A}^*$.

RNNAs (Schröder, Kozen, M, Wißmann 2017)

Regular nominal non-deterministic automata (RNNAs) are coalgebras $Q \to \{0,1\} \times \mathcal{P}_{\mathsf{ufs}}(\mathbb{A} \times Q) \times \mathcal{P}_{\mathsf{ufs}}([\mathbb{A}]Q)$.

Concretely: $\bar{\mathbb{A}} = \mathbb{A} \cup \{|a : a \in \mathbb{A}\}$; ordinary and binding transitions

Every state accepts its bar language $L_q \subseteq \bar{\mathbb{A}}^*/{=_\alpha}$.



equivalence generated by $x|av =_\alpha x|bw$ $\forall a, b \in \mathbb{A}$ and $x, v, w \in \bar{\mathbb{A}}^*$ s.th. $\langle a \rangle v = \langle b \rangle w$.

Problem: $\mathcal{P}_{\mathsf{fs}}X$ and $\mathcal{P}_{\mathsf{ufs}}X$ are no complete lattices (not even ω-cpos)! ☹

Theorem ☺

The categories $\mathsf{Kl}(\mathcal{P}_{\mathsf{fs}})$ and $\mathsf{Kl}(\mathcal{P}_{\mathsf{ufs}})$ are left strictly $\mathsf{DCPO}_{\perp}$-enriched.

Which functors extend to $\mathsf{Kl}(\mathcal{P}_{\mathsf{fs}})$ and $\mathsf{Kl}(\mathcal{P}_{\mathsf{ufs}})$?

Extending Functors to $\mathsf{Kl}(\mathcal{P}_{\mathsf{fs}})$ and $\mathsf{Kl}(\mathcal{P}_{\mathsf{ufs}})$

Definition

Binding polynomial functors on **Nom** are given by the grammar

$$F ::= C \mid \mathsf{Id} \mid [\mathbb{A}](-) \mid F \times F \mid \coprod_{i \in I} F_i \mid FF,$$

where C ranges over constant functors and I is an index set.

Theorem

1. Every binding polynomial functor F has a canonical locally monotone extension $\bar{F}$ on $\mathsf{Kl}(\mathcal{P}_{\mathsf{ufs}})$. $\rightsquigarrow \mu F =$ terminal $\bar{F}$-coalgebra

without $[\mathbb{A}](-)$

2. Every polynomial functor F has a canonical locally monotone extension $\bar{F}$ on $\mathsf{Kl}(\mathcal{P}_{\mathsf{fs}})$. $\rightsquigarrow \mu F =$ terminal $\bar{F}$-coalgebra

Salient Point

- There is a canonical natural isomorphism $\psi_X \colon \mathcal{P}_{\mathsf{fs}}([\mathbb{A}]X) \to [\mathbb{A}]\mathcal{P}_{\mathsf{fs}}X$.
- Its inverse is not a distributive law $[\mathbb{A}](-)$ over $\mathcal{P}_{\mathsf{fs}}$.

Proposition

The abstraction functor admits a (functor over monad) distributive law

$$[\mathbb{A}](\mathcal{P}_{\mathsf{ufs}}X) \xrightarrow{\varrho_X} \mathcal{P}_{\mathsf{ufs}}([\mathbb{A}]X) \qquad \langle a\rangle S \longmapsto \{\langle a\rangle s : s \in S\}$$

Quotient of the canonical distributive law

$$\begin{array}{ccc} \mathbb{A} \times \mathcal{P}_{\mathsf{ufs}}X & \xrightarrow{\lambda_X} & \mathcal{P}_{\mathsf{ufs}}(\mathbb{A} \times X) \\ {\scriptstyle q_{\mathcal{P}_{\mathsf{ufs}}X}} \downarrow & & \downarrow {\scriptstyle \mathcal{P}_{\mathsf{ufs}} q_X} \\ [\mathbb{A}](\mathcal{P}_{\mathsf{ufs}}X) & \xrightarrow{\varrho_X} & \mathcal{P}_{\mathsf{ufs}}([\mathbb{A}]X) \end{array} \qquad \mathbb{A} \times X \xrightarrow{q_X} [\mathbb{A}]X$$

Note: ϱ_X is not well-defined for $\mathcal{P}_{\mathsf{fs}}$; the inverse of ψ_X is more involved.

Instance 1: NOFAs

- NOFAs are coalgebras $Q \xrightarrow{c} \{0,1\} \times \mathcal{P}_{\mathsf{fs}}(\mathbb{A} \times Q) \cong \mathcal{P}_{\mathsf{fs}}(1 + \mathbb{A} \times X)$
- Take $FX = 1 + \mathbb{A} \times X$ and $T = \mathcal{P}_{\mathsf{fs}}$.
- We have a locally monotone extension $\bar{F} \colon \mathsf{Kl}(\mathcal{P}_{\mathsf{fs}}) \to \mathsf{Kl}(\mathcal{P}_{\mathsf{fs}})$.
- Its terminal coalgebra is $\mu F = \mathbb{A}^*$.
- The coalgebraic trace map $Q \xrightarrow{\mathsf{tr}_c} \mathcal{P}_{\mathsf{fs}}(\mathbb{A}^*)$ satisfies

$$\mathsf{tr}_c(q) = \text{data language of state } q.$$

Instance 2: RNNAs

- RNNAs are coalgebras
  $Q \xrightarrow{c} \{0,1\} \times \mathcal{P}_{\mathsf{ufs}}(\mathbb{A} \times X) \times \mathcal{P}_{\mathsf{ufs}}([\mathbb{A}]X) \cong \mathcal{P}_{\mathsf{ufs}}(1 + \mathbb{A} \times X + [\mathbb{A}]X)$

- Take $FX = 1 + \mathbb{A} \times X + [\mathbb{A}]X$ and $T = \mathcal{P}_{\mathsf{ufs}}$.

- We have a locally monotone extension $\bar{F} \colon \mathsf{Kl}(\mathcal{P}_{\mathsf{ufs}}) \to \mathsf{Kl}(\mathcal{P}_{\mathsf{ufs}})$.

- Its terminal coalgebra is $\mu F = \overline{\mathbb{A}}^* / {=_\alpha}$. (by Gabbay and Pitts)

- The coalgebraic trace map $Q \xrightarrow{\mathsf{tr}_c} \mathcal{P}_{\mathsf{ufs}}(\overline{\mathbb{A}}^* / {=_\alpha})$ satisfies

$$\mathsf{tr}_c(q) = \text{bar language of state } q.$$

Summary

- Revisited Kleisli style coalgebraic trace semantics:
  initial algebra μF extends to terminal $\bar{F}$-coalgebra on $\mathsf{Kl}(T)$
  (finitariness or convergence of initial-algebra chain are **not** needed)

- Language semantics of NOFAs and RNNAs are instances of Kleisli style coalgebraic trace semantics

Further Work

Further Work (in the paper):

- EM-style coalgebraic language semantics of NOFAs and RNNAs using extension semantics by Jacobs, Silva, Sokolova

Subtleties: Lifting $[\mathbb{A}](-)$ to $\mathsf{EM}(\mathcal{P}_{\mathsf{ufs}})$.
(Once again our proof does not work for $\mathcal{P}_{\mathsf{fs}}$.)

$$\mathcal{P}_{\mathsf{fs}}(\mathbb{A} \times X) \cong (\mathcal{P}_{\mathsf{fs}}X)^{\mathbb{A}} \qquad \text{but} \qquad \mathcal{P}_{\mathsf{ufs}}(\mathbb{A} \times X) \overset{\ncong}{\rightarrowtail} (\mathcal{P}_{\mathsf{ufs}}X)^{\mathbb{A}}$$

Result: The coalgebraic language semantics arising from generalized determinization yields the data languages of NOFAs and the bar language of RNNAs, respectively.

Proof: use extension natural transformations $\varepsilon\colon TF \to GT$

$$\mathcal{P}_{\mathsf{fs}}(1 + \mathbb{A} \times X) \xrightarrow{\cong} \{0,1\} \times (\mathcal{P}_{\mathsf{fs}}X)^{\mathbb{A}}$$

$$\mathcal{P}_{\mathsf{ufs}}(1 + \mathbb{A} \times X + [\mathbb{A}]X) \rightarrowtail \{0,1\} \times (\mathcal{P}_{\mathsf{ufs}}X)^{\mathbb{A}} \times [\mathbb{A}](\mathcal{P}_{\mathsf{ufs}}X)$$

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Further work (in the future):

- Coalgebraic methods/techniques applied to NOFAs, RNNAs, and other nominal systems, e.g.
- coalgebraic ε-elimination
- coalgebraic up-to-techniques might lead to new proof principles and algorithms
- semantics for nominal systems based on graded monads should lead to a nominal spectrum of system equivalences
  (generalizing the van Glabbeek linear time – branching time spectrum)