Algebraic Presentation of Semifree Monads

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Overview

- Preliminary definitions
- Motivation & Goal
- The *semifree monad* M^{s} on a monad M
- The general presentation of M^{s}
- Examples
- Relationship to Ideal monads and Monad transformers
- Conclusion & Future work

Monads

Monads are $(M : \mathsf{C} \to \mathsf{C}, \eta : \mathrm{id} \Rightarrow M, \mu : MM \Rightarrow M)$ satisfying

$$\begin{array}{ccc} M & \xrightarrow{M\eta} & M^2 \\ {\scriptstyle \eta M}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu} \\ M^2 & \xrightarrow[\mu]{} & M \end{array} \qquad\qquad \begin{array}{ccc} M^3 & \xrightarrow{\mu M} & M^2 \\ {\scriptstyle M\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ M^2 & \xrightarrow[\mu]{} & M \end{array}$$

M-algebras are $(X, \alpha : MX \to X)$ satisfying (1)-(2). Category: $\mathbf{EM}(M)$

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & MX \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \alpha} \\ & & X \end{array} \qquad (1) \qquad\qquad \begin{array}{ccc} M^2X & \xrightarrow{\mu_X} & MX \\ {\scriptstyle M\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ MX & \xrightarrow[\alpha]{} & X \end{array} \qquad (2)$$

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\qquad\qquad
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M-semialgebras must satisfy only (2). Category: $\mathbf{EM}_s(M)$.

$$
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 & & X
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\qquad (1)
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\begin{array}{ccc}
M^2X & \xrightarrow{\mu_X} & MX \\
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MX & \xrightarrow[\alpha]{} & X
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$$

Distributive law

Distributive laws of T over M are $\lambda : MT \Rightarrow TM$ s.t (3)-(6) commute.

$$\begin{array}{ccc} & T & \\ {\scriptstyle \eta^M T} \swarrow & & \searrow {\scriptstyle T\eta^M} \\ MT & \xrightarrow[\lambda]{} & TM \end{array} \tag{3}$$

$$\begin{array}{ccc} & M & \\ {\scriptstyle M\eta^T} \swarrow & & \searrow {\scriptstyle \eta^T M} \\ MT & \xrightarrow[\lambda]{} & TM \end{array} \tag{4}$$

$$\begin{array}{ccccc} MMT & \xrightarrow{M\lambda} & MTM & \xrightarrow{\lambda M} & TMM \\ {\scriptstyle \mu^M T}\downarrow & & & & \downarrow{\scriptstyle T\mu^M} \\ MT & & \xrightarrow[\lambda]{} & & TM \end{array} \tag{5}$$

$$\begin{array}{ccccc} MTT & \xrightarrow{\lambda T} & MTM & \xrightarrow{T\lambda} & TMM \\ {\scriptstyle M\mu^T}\downarrow & & & & \downarrow{\scriptstyle \mu^T M} \\ MT & & \xrightarrow[\lambda]{} & & TM \end{array} \tag{6}$$

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Weak distributive laws only require (4)-(6).

$$\begin{array}{ccc} & T & \\ {}^{\eta^M T}\swarrow & & \searrow^{T\eta^M} \\ MT & \xrightarrow[\lambda]{} & TM \end{array} \qquad (3)$$

$$\begin{array}{ccc} & M & \\ {}^{M\eta^T}\swarrow & & \searrow^{\eta^T M} \\ MT & \xrightarrow[\lambda]{} & TM \end{array} \qquad (4)$$

$$\begin{CD} MMT @>{M\lambda}>> MTM @>{\lambda M}>> TMM \\ @V{\mu^M T}VV @. @VV{T\mu^M}V \\ MT @>>{\lambda}> @. TM \end{CD} \qquad (5)$$

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Algebraic theories

Algebraic theories are (Σ, E), where

- $\Sigma = \{f, g, \ldots\}$ set of operations symbols with arity
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Adjunction: $\mathrm{Set} \underset{U}{\overset{\mathrm{Free}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Alg}(\Sigma, E)$ where $\mathrm{Free}X = {}^{\mathcal{T}(\Sigma,X)}/_{E}$.
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(Σ, E) is an *algebraic presentation* of M if

$$T_{\Sigma,E} \cong M \qquad \text{or equivalently} \qquad \mathbf{EM}(M) \cong_{\mathrm{conc}} \mathbf{Alg}(\Sigma, E)$$

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Petrișan & Sarkis'21

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Subclass of distr. laws T over M^{s} $\underset{(*)}{\Longleftrightarrow}$ Subclass of liftings $\tilde{T}$ on $\mathbf{EM}(M^{\mathrm{s}})$

Goal

Question: Given algebraic presentation of M, can we deduce an algebraic presentation of M^{s}?

Theorem
For (Σ, E) an algebraic presentation of (necessarily finitary) monad (M, η, μ). Then $(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$ is an algebraic presentation of $(M^{\mathrm{s}}, \eta^{\mathrm{s}}, \mu^{\mathrm{s}})$.

Presentations of monads by algebraic theories

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Theory of *monoids* $\Sigma := \{e^{(0)}, \cdot^{(2)}\}$ $E := \left\{ \begin{array}{c} v \cdot e = v, \\ e \cdot v = v, \\ (u \cdot v) \cdot w = u \cdot (v \cdot w) \end{array} \right\}$	*List* monad $(-)^*$
Theory of *convex sets* $\Sigma := \{+_p : 2 \mid p \in (0,1)\}, E :=$ $\left\{ \begin{array}{c} v +_p v = v, \\ u +_p v = v +_{1-p} u, \\ (u +_q v) +_p w = u +_{pq} (v +_{p(1-q)/1-pq} w) \end{array} \right\}$	*Distribution* monad $\mathcal{D}$

Semifree monads

For monad M, Petrișan & Sarkis'21 introduced the *semifree monad* M^{s}:

$$
\begin{aligned}
M^{\mathrm{s}} &:= \mathsf{Id}_C + M, \\
\eta^{\mathrm{s}} &:= \mathsf{inl}^{\mathsf{Id}+M}, \\
\mu^{\mathrm{s}} &:= [\mathsf{Id}_{\mathsf{Id}+M}, \mathsf{inr}^{\mathsf{Id}+M} \circ \mu \circ M[\eta, \mathsf{Id}_M]].
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Lemma: $\mathbf{EM}(M^{\mathrm{s}}) \cong_{\mathsf{conc}} \mathbf{EM}_s(M)$.

Lemma: $\{\lambda : MT \overset{\text{w.d.l}}{\Rightarrow} TM\} \overset{\text{bijection}}{\leftrightarrow} \{\delta : M^{\mathrm{s}} T \overset{\text{d.l}}{\Rightarrow} TM^{\mathrm{s}} \text{ satisf. extra condition}\}$.

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  Notice:

$$
\begin{aligned}
(\alpha \cdot \eta_X) \cdot (\alpha \cdot \eta_X) &= \alpha \cdot M\alpha \cdot \eta_{MX} \cdot \eta_X && (\eta \text{ nat.}) \\
&= \alpha \cdot \mu_X \cdot \eta_{MX} \cdot \eta_X && (\text{assoc. } \alpha) \\
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- This idempotent will be the interpretation of (a : 1).

Contribution

Definition
Given an algebraic theory (Σ, E), define the algebraic theory $(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$ by $\Sigma^{\mathrm{s}} := \Sigma \uplus \{\mathsf{a} : 1\}$ and E^{s} by:

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Theorem
Given (Σ, E) an algebraic presentation of a finitary Set*-monad (M, η, μ). Then $(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$ is an algebraic presentation of $(M^{\mathrm{s}}, \eta^{\mathrm{s}}, \mu^{\mathrm{s}})$.*

Sketch of proof

It suffices to prove it for the free monad $T = T_{\Sigma,E}$. Since $\mathbf{EM}(T^{\mathrm{s}}) \cong_{\mathsf{conc}} \mathbf{EM}_s(T)$, it suffices to prove $\mathbf{EM}_s(T) \cong_{\mathsf{conc}} \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$.

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$$\begin{cases} G : \mathbf{EM}_s(T) \to \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}}) \\ \quad G(X, \alpha) := (X, \langle\!|\cdot|\!\rangle) \\ \quad G(f) := f \end{cases}$$

$$\langle\!|\mathsf{a}|\!\rangle := \left(X \xrightarrow{\eta_X} TX \xrightarrow{\alpha} X \right),$$

$$\langle\!|\mathsf{op}|\!\rangle := \left(X^n \xrightarrow{(\eta_X)^n} (TX)^n \xrightarrow{[\![\mathsf{op}]\!]^X} TX \xrightarrow{\alpha} X \right),$$

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Lemma: G and H are well-defined and inverses. □

Examples

- Maybe: $(- + 1)^{\mathrm{s}}$ presented by

$$\Sigma^{\mathrm{s}} = \{\bullet : 0, \mathsf{a} : 1\}, E^{\mathrm{s}} = \{\mathsf{aa}v = \mathsf{a}v,\ \mathsf{a}\bullet = \bullet\}.$$

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Lemma: The terms of depth $\geqslant 1$ satisfy:

$$\mathsf{a}t(v_1, \dots, v_n) = t(v_1, \dots, v_n), \text{ and}$$
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Examples 2

- Monoid: $((-)^*)^{\mathrm{s}}$ presented by $\Sigma_*^{\mathrm{s}} = \{e : 0, \mathsf{a} : 1, \cdot : 2\}$

$$E_*^{\mathrm{s}} = \left\{ \begin{array}{cc} \mathsf{a}\mathsf{a}v = \mathsf{a}v, & e \cdot v = \mathsf{a}v, \\ \mathsf{a}e = e, & v \cdot e = \mathsf{a}v, \\ \mathsf{a}(u \cdot v) = u \cdot v, & (u \cdot v) \cdot w = u \cdot (v \cdot w) \\ \mathsf{a}u \cdot \mathsf{a}v = u \cdot v, & \end{array} \right\}.$$

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$$E^{\mathrm{s}}_* = \left\{ \begin{matrix} e \cdot e = e, & e \cdot v = v \cdot e, \\ \\ e \cdot (u \cdot v) = u \cdot v, & (u \cdot v) \cdot w = u \cdot (v \cdot w) \\ \end{matrix} \right\}.$$

Algebras look like semigroups S with a retract $R \subseteq S$ that contains $S \cdot S$, is a monoid, and with retraction $e \cdot - : S \to S$.

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- $\mathsf{Id}^{\mathrm{s}^n}(X) = X + \ldots + X \cong X \times (n+1)$ presented by n unary idempotents $\Sigma^{\mathrm{s}^n} = \{\mathsf{a}_1, \ldots, \mathsf{a}_n : 1\}$, and an idempotent a_i absorbs another one a_j on both sides whenever $i < j$:

$$E^{\mathrm{s}^n} = \{\mathsf{a}_i\mathsf{a}_j v = \mathsf{a}_{\min(i,j)} v \mid 1 \leqslant i, j \leqslant n\}.$$

Relationship to Ideal monads and Monad Transformers

Definition: In category C with finite coproducts, an *ideal monad* is (T, η, μ, T_0, m_0) where (T, η, μ) is a monad with

- functor $T = \mathsf{Id} + T_0$
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- μ "restricts" to m_0

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Lemma: Semifree monads are *ideal monads*

Is $(-)^{\mathrm{s}}$ a monad transformer? No, but

Lemma: $(-)^{\mathrm{s}}$ is not pointed, but co-pointed endofunctor on **Mon**(C). It is in fact a comonad on **Mon**(C).

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- Presentation of $M \Rightarrow$ presentation of $M(- + 1)$ is adding a constant and no equations.
- Presentation of M and T and existence of a distributive law $\lambda : MT \Rightarrow TM$. Zwart & Marsden (LICS 2009): presentation of TM can be inferred.

Future work

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- Infer presentation of coproduct of monads Luth & Ghani'02?
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- Infer presentations in other categories from the ones in Set (e.g. Mio et al'21)?
- No-go theorems for weak distributive laws, analogous to Zwart & Marsden (LICS 2019)? Zwart conjectures that reduction-to-variable axioms are needed. Conjecture: weak distributive laws always exist? Goal: understand better what $(-)^{\mathrm{s}}$ tells us about existence of weak distributive laws.

Thanks

Thank you for your attention !