Corecursion up-to via Causal Transformations

Damien Pous[1] Jurriaan Rot[2] Ruben Turkenburg[2]

[1]Plume, LIP, CNRS, ENS de Lyon, Université de Lyon, UMR 5668, France

[2]Radboud University, Nijmegen, Netherlands

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Outline

Corecursion & Coinduction

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \zeta} \\ BX & \xrightarrow{Bf} & BZ \end{array} \tag{1}$$

- Using finality for definitions and proofs

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- Using finality for definitions and proofs
- When does this work? Always, by definition of finality

Pointwise addition of streams

- Streams: represented by coalgebras for the functor $A \times (-)$
- Final coalgebra: A^ω
- Defining pointwise addition of real streams:

$$\begin{array}{ccc} \mathbb{R}^\omega \times \mathbb{R}^\omega & \xrightarrow{\quad \oplus \quad} & \mathbb{R}^\omega \\ {\scriptstyle \langle p, r \rangle} \downarrow & & \downarrow {\scriptstyle \langle h, t \rangle} \\ \mathbb{R} \times \mathbb{R}^\omega \times \mathbb{R}^\omega & \xrightarrow{\mathrm{id}_{\mathbb{R}} \times \oplus} & \mathbb{R} \times \mathbb{R}^\omega \end{array} \tag{2}$$

$$p(x, y) = h(x) + h(y) \qquad (x \oplus y)_0 = x_0 + y_0 \tag{3}$$

$$r(x, y) = (t(x), t(y)) \qquad (x \oplus y)' = x' \oplus y' \tag{4}$$

Shuffle product of streams

- Given by the equations:

$$(x \otimes y)_0 = x_0 \times y_0 \tag{5}$$
$$(x \otimes y)' = (x \otimes y') \oplus (x' \otimes y) \tag{6}$$

- Not a coalgebra for $\mathbb{R} \times (-)$! Applying $\oplus$ to streams not yet defined
- But there is a unique solution in $\mathbb{R}^\omega$
- How do we find it? Up-to techniques!

Corecursion up-to an algebra

$$
\begin{array}{ccccc}
X & \xrightarrow{\quad f^a \quad} & & & Z \\
{\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle \zeta} \\
BFX & \xrightarrow{BFf^a} & BFZ & \xrightarrow{Ba} & BZ
\end{array}
\qquad
\begin{array}{c}
FZ \\
\downarrow{\scriptstyle a} \\
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\end{array}
\tag{7}
$$

- Valid corecursion up-to a: for all (X, f), unique f^a giving commutativity.

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Corecursion up-to an algebra

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- Valid corecursion up-to a: for all (X, f), unique f^a giving commutativity.
- $F = \text{id}$ and $a = \text{id}$: plain corecursion
- Shuffle product: $B = \mathbb{R} \times (-)$ and $F = (-)^2$
  - $f(x, y) = (x_0 \times y_0, ((x, y'), (x', y)))$
  - $a = \oplus$

When do we have validity?

- Two existing sufficient conditions:

  1. a is induced by a distributive law $\lambda\colon FB \Rightarrow BF$ and the base category has countable coproducts (or F is a monad and λ a distributive law of this monad over the functor B);
  2. B is a polynomial set functor, and $a\colon FB_\omega \to B_\omega$ is a causal F-algebra.

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     - Causality for streams: require at most n elements of the input to produce n elements of the output
- In our paper:
  3. Existence of a causal transformation $\alpha\colon F\overline{B} \Rightarrow \overline{B}$ for which a is the κ-th component α_κ.

Delaying Trees

- Finite powerset functor $\mathcal{P}_f$, final coalgebra $(\mathcal{T}_f, c)$: finitely branching strongly extensional trees, c gives set of children

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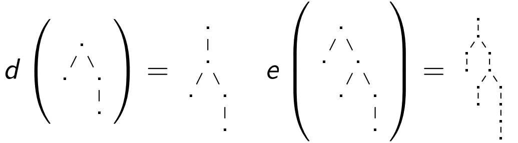
- Delay function d adds unary node to top of tree
- Define the function e delaying all nodes of a (possibly infinite depth) tree

$$c(e(t)) = \mathcal{P}_f(d \circ e)(c(t))$$

$$\begin{array}{ccccc} \mathcal{T}_f & & \xrightarrow{\quad e \quad} & & \mathcal{T}_f \\ {\scriptstyle c}\downarrow & & & & \downarrow{\scriptstyle c} \\ \mathcal{P}_f\mathcal{T}_f & \xrightarrow{\mathcal{P}_f e} & \mathcal{P}_f\mathcal{T}_f & \xrightarrow{\mathcal{P}_f d} & \mathcal{P}_f\mathcal{T}_f \end{array}$$



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- If B has a companion: Causal transformation $\implies$ Distributive law
- B an ω-continuous set functor: Causal algebra $\iff$ Causal transformation
- Polynomial set functors: Distributive law $\iff$ Causal algebra $\iff$ Causal transformation

Outline

Corecursive Algebras

- Unique f' giving commutativity of:

$$\begin{array}{ccc} X & \xrightarrow{f'} & A \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle a} \\ BX & \xrightarrow{Bf'} & BA \end{array}$$

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- Each connecting map in final sequence forms part of an algebra

The main result

- At each stage of the final sequence of B

$$\begin{array}{ccccc}
X & & \xrightarrow{\quad f_k^{\alpha} \quad} & & \overline{B}(k) \\
\downarrow{\scriptstyle f} & & & & \uparrow{\scriptstyle \overline{B}(k+1,k)} \\
BFX & \xrightarrow{BFf_k^{\alpha}} & BF\overline{B}(k) & \xrightarrow{B\alpha_k} & B\overline{B}(k)
\end{array}$$

- The BF-algebra $\overline{B}(k+1,k) \circ B\alpha_k$ is corecursive
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- For $F = \mathrm{id}$ and $\alpha = \mathrm{id}$, this recovers an observation due to (we believe) Paul Levy
- Corecursion up-to even without a final coalgebra!
- With a final coalgebra: valid corecursion up-to a as before

Proof sketch

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- Transfinite induction on k
- Base case is unique map into 1
- Successor: $f_{j+1}^\alpha \triangleq B\alpha_j \circ BFf_j^\alpha \circ f$
- Limit: Show that maps $f_j^\alpha \colon X \to \overline{B}(j)$ for $j < k$ form a cone

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\overline{\mathcal{P}}(\omega) & & \xrightarrow{\quad e \quad} & & \overline{\mathcal{P}}(\omega) \\
{\scriptstyle c}\big\downarrow & & & & \big\uparrow{\scriptstyle \overline{\mathcal{P}}(\omega+1,\omega)} \\
\mathcal{P}\overline{\mathcal{P}}(\omega) & \xrightarrow{\mathcal{P}e} & \mathcal{P}\overline{\mathcal{P}}(\omega) & \xrightarrow{\mathcal{P}d} & \mathcal{P}\overline{\mathcal{P}}(\omega)
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- Weaker than earlier definition

Bartels (2003)

- We recover:

Corollary

For $\lambda \colon FB \Rightarrow BF$ a distributive law, if $\overline{B}$ stabilizes at κ, then corecursion up-to the algebra induced by λ on $\overline{B}(\kappa)$ is valid.

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For $\lambda\colon FB \Rightarrow BF$ a distributive law, if $\overline{B}$ stabilizes at κ, then corecursion up-to the algebra induced by λ on $\overline{B}(\kappa)$ is valid.

- Compare:

Theorem (Bartels)

For $\lambda\colon FB \Rightarrow BF$ a distributive law, if either

- the category $\mathcal{C}$ has countable coproducts, or
- F is a monad and λ is a distributive law of F

then corecursion up-to the algebra induced by λ on the final coalgebra of B is valid.

Outline

The category $\mathcal{K}$

Definition

For functors $F\colon \mathcal{C} \to \mathcal{D}$, $A\colon \mathcal{C} \to \mathcal{C}$ and $B\colon \mathcal{D} \to \mathcal{D}$, a causal transformation from A to B is a natural transformation $\alpha\colon F\overline{A} \Rightarrow \overline{B}$.

- Note: A and B may now be different!

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Theorem

We have a category $\mathcal{K}$ with the following data:

- objects are pairs $(A, \mathcal{C})$ with $A\colon \mathcal{C} \to \mathcal{C}$ a functor on a complete category $\mathcal{C}$.
- morphisms from $(A, \mathcal{C})$ to $(B, \mathcal{D})$ are pairs (F, α) with $\alpha\colon F\overline{A} \Rightarrow \overline{B}$.

Some constructions on $\mathcal{K}$

- Operations and basic building blocks, already known for distributive laws [Bonchi et al., 2016]

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The category $\mathcal{K}$ has all products.

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Theorem

The category $\mathcal{K}$ has all products.

- We have morphisms for constant and coproduct functors

Example

Corecursion up-to for unary operation f on streams:

$$f(x)_0 = x_0 \quad f(x)' = (x \oplus f(x')) \oplus f(x'') \tag{9}$$

Take functor $F(F(\mathbb{R}^\omega + \mathrm{Id}) + \mathrm{Id})$ and build algebra $\oplus \circ [\oplus \circ [\mathrm{id}; \mathrm{id}]; \mathrm{id}]$ on $\mathbb{R}^\omega$

The 2-category $\mathcal{K}$

Definition

Given two causal transformations $(F, \alpha), (G, \beta) \colon A \dot{\to} B$, an arrow from (F, α) to (G, β) is a natural transformation $\kappa \colon F \Rightarrow G$ such that $\beta \circ \kappa\overline{A} = \alpha$. These arrows turn $\mathcal{K}$ into a 2-category.

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- Constructions distributive laws $\leftrightarrow$ causal transformations as a 2-functor?

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Proofs

- Already known [Bonchi et al., 2014]:
  - Contextual closure is compatible, gives valid up-to technique
  - Contextual closure commutes with lifting of coalgebra functor

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  - Contextual closure is compatible, gives valid up-to technique
  - Contextual closure commutes with lifting of coalgebra functor
- Here:
  - Instantiating results to lattices gives causal transformations known from other recent work
  - Causal transformation for contextual closure
  - Apply earlier results in the fibrational setting where fibres are complete lattices: coinduction up-to
  - Requirements: $(B, \mathbb{B})$ fibration map and $\alpha\colon F\overline{B} \Rightarrow \overline{B}$ lifts to $\mathbb{F}\overline{\mathbb{B}} \Rightarrow \overline{\mathbb{B}}$

Fibration map

- Fibration map requirement in relation lifting case: needs weak pullback preservation by B
- $F = \mathsf{Id}, \alpha = \mathsf{id}, \mathbb{F}(R) =$ least equivalence relation containing R

$$B(X) = \{(x, y, z) \mid |\{x, y, z\}| \leq 2\} \tag{10}$$

- α lifts but up-to-equivalence is not sound for B
- Consequence: not all causal transformations definable by a distributive law

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Conclusion

- Causal transformations give a nice foundation for corecursion/coinduction up-to (although not constructive)
- More general than causal algebras: ω-continuity not required
- Some causal transformations not definable by a distributive law
- A form of corecursion up-to for functors without a final coalgebra
- Desirable properties enjoyed by distributive laws still hold:
  - Compositionality
  - Application to lattices give up-to techniques for coinductive proofs