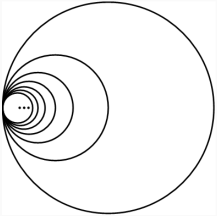
Nominal Topology for Data Languages



Henning Urbat
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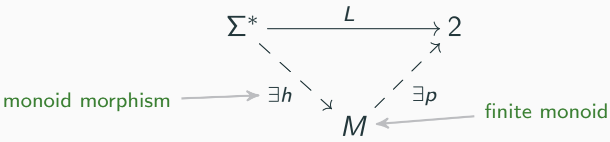
CMCS 2022

Regular Languages

Algebraic perspective: Regular = monoid-recognizable languages.



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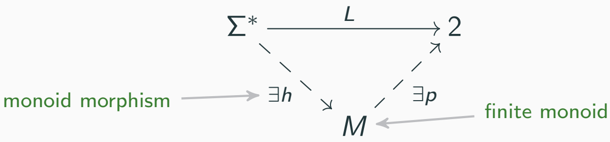

Topological perspective: Profinite words.

$$\widehat{\Sigma^*} \;=\; \text{limit of the canonical cofiltered diagram}$$

$$D\colon \Sigma^*{\downarrow}\mathbf{Mon_f} \to \mathbf{Set_f}$$

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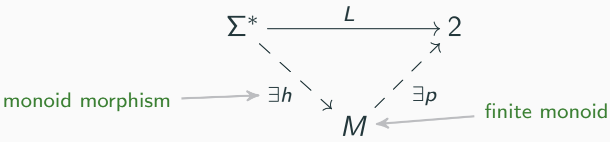

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free completion under cofiltered limits

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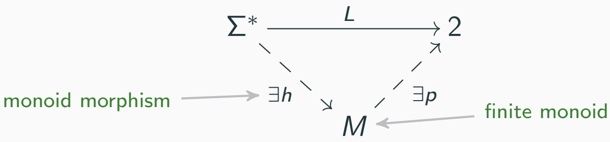

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$$D\colon \Sigma^*{\downarrow}\mathbf{Mon_f} \to \mathbf{Set_f} \rightarrowtail \mathrm{Pro}(\mathbf{Set_f}) = \mathbf{Stone}$$

compact + Hausdorff + base of clopens

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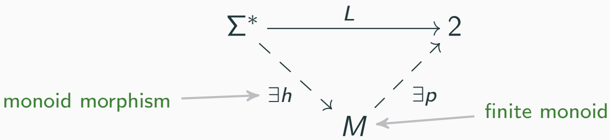

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Theorem (Topological Characterization of Regularity)

$$\text{Regular languages } \Sigma^* \to 2 \quad \cong \quad \text{continuous predicates } \widehat{\Sigma^*} \to 2.$$

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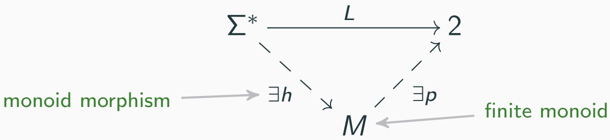

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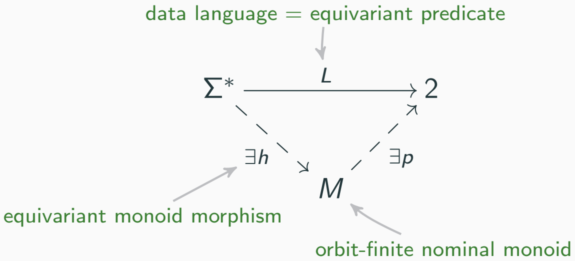
Data Languages?

Data Languages

- Nominal sets: sets X equipped with $\mathrm{supp}\colon X \to \mathcal{P}_{\mathsf{f}}(\mathbb{A})$.
- Recognizable data languages over $\Sigma \in \mathbf{Nom}$:



Equivalent descriptions: rigid MSO and single-use register automata.

Bojańczyk 2008, Colcombet, Ley & Puppis 2011, Bojańczyk & Stefański 2020

Data Languages: Topological Perspective

For regular languages:

$$\mathrm{Pro}(\mathbf{Set_f}) \quad = \quad \mathbf{Stone}.$$

For recognizable data languages:

$$\mathrm{Pro}(\mathbf{Nom_{of}}) \quad = \quad \text{Nominal Stone spaces ? ? ?}$$

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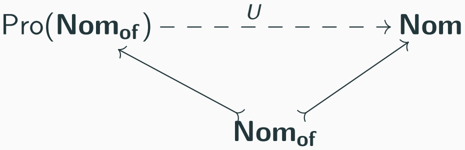

☹ U not faithful!

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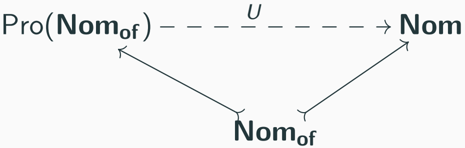

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Bounded Nominal Stone Spaces

A nominal set X is *n*-bounded if $|\mathrm{supp}(x)| \leq n$ for all $x \in X$.

Theorem

$$\mathrm{Pro}(\mathbf{Nom_{of,n}}) \quad = \quad n\text{-bounded nominal Stone spaces.}$$

↑ *n*-bounded orbit-finite sets

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↑ n-bounded orbit-finite sets

A nominal Stone space is a nominal topological space that is

- **compact:** every open cover has an orbit-finite subcover;
- **Hausdorff**: any $x \not\equiv_S y$ separated by disjoint S-supported open sets;
- **zero-dimensional**: base of sets $f^{-1}[d]$ where $f : X \to D$ and $d \in D$.

↑ orbit-finite + discrete

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orbit-finite + discrete

$\neq$ **Gabbay, Litak, Petrişan 2011**

Further Results

- Spaces of bounded pro-orbit-finite words.
- Recognizable data languages as continuous predicates.
- Equational theory using pro-orbit-finite equations.
  (joint work with F. Birkmann & S. Milius)
- Duality theory based on nominal Stone duality:

(n-bounded nominal Stone spaces)$^{\mathsf{op}}$

$$\simeq$$

locally n-atomic orbit-finitely complete nominal boolean algebras.