Automata and coalgebras in categories of species*

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Abstract. We study generalized automata (in the sense of Adámek-Trnková) in Joyal's category of (set-valued) combinatorial species, and as an important preliminary step, we study coalgebras for its derivative endofunctor ∂ and for the 'Euler homogeneity operator' $L \circ \partial$ arising from the adjunction $L \dashv \partial \dashv R$.

1 Introduction

The theory of combinatorial species arose in the work of André Joyal [71,72] as categorification of the theory of generating functions [130]; crafting a bijective proof [96] to grok numerical identities in terms of bijections between finite sets is acknowledged as *the* fundamental problem in modern combinatorics. For Joyal, a 'species of structure' is a functor having domain the category of finite sets and bijections; properties of the category of all such functors can then be given combinatorial meaning, and combinatorial identities acquire meaning as bijective proofs (=isomorphisms of functors). Joyal's insighftul proof of Cayley's counting of trees [19], paved the way to a booming development of techniques (propelled by the support of an insider of enumerative combinatorics, and genius, as C.G. Rota) in domains such as representation theory of groups [20,83,111,133], the study of set partitions [17,68,95], Möbius functions [96,114,115], graph theory [100], up to the exciting field of *combinatorial differential equations* [82,83,97,12]. This wealth of applications is by no means limited to the field of enumerative combinatorics; the operation of *plethystic substitution* [10,104,105] is recognized as the fundamental building block in the theory of *operads* envisioned by J.P. May [93,94] and finds applications to algebraic topology and algebraic geometry [30,36,84,106], logic and computer science [32,33,135], theoretical physics [37,38], and more.

At about the same time, another application of category theory gained momentum: the idea of interpreting *abstract state machines* inside general categories. The line of research initiated by Arbib–Manes [5,109], Goguen [43,44,45], Naudé [101,102], and others [39,58,63,126] culminated into Ehrig's monograph [25] on automata 'valued' in an abstract monoidal category $\mathcal{K}$. This provides a systematic, category–theoretic insight into the transition from determinism to non-determinism, that can be seen as the passage from automata in a monoidal category [98], to automata in the Kleisli category

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of an opmonoidal monad [51,65] (such as for example the probability distribution monads for convex spaces, [24,31,64,66,92] or one of its companions –the subdistribution or unnormalized distribution monad).

The category-theoretic content of such an approach to 'machines' goes a long way: a tentative chronology follows, but it can only scratch the surface of an immense, often submerged, body of research.

- [1,3] introduced the notion of an *F-automaton* in order to abstract even further from the monoidal case the 'dynamics' igniting the behaviour of an abstract machine; the progression in abstraction is as follows: from Cartesian machines, i.e. spans $E \leftarrow A \times B \to B$, one goes to monoidal ones, i.e. spans $E \leftarrow A \otimes B \to B$; these are the objects of categories $\mathsf{Mly}_{(\mathscr{K},\otimes)}(A, B)$. Subsequently, one abstracts the action of $A \otimes \_$ on E even further, using a generic endofunctor $F : \mathscr{K} \to \mathscr{K}$; this is the category $\mathsf{Mly}_{\mathscr{K}}(F, B)$.
- Only few years prior, extensive work of Betti-Kasangian [13,14] and Kasangian-Rosebrugh [74] pushed for the adoption of 'profunctorial' models for automata, capable to pinpoint their behaviour, and their minimization, as a universal property [42,45].
- An insightful idea of Katis, Sabadini and Walters [75,76] recognized that categories of automata organize themselves as the hom-categories of a bicategory $\mathsf{KSW}(\mathscr{K})$.[1]
- in [48,51] René Guitart introduces the bicategory Mac as a refinement of a bicategory of spans.[2] In [56], Guitart proves Mac is simply the Kleisli bicategory of the 2-monad of cocompletion under lax colimits. This theme is reprised in [50] where Guitart introduces the notion of *lax coend* [60,85] as a technical preliminary to expand on the theme of [56].

Pushing further these ideas intersects the most prolific branches of modern category theory.

Building on [25], R. Paré proposed in [107] the notion of a *Mealy morphism* as a proxy between strong functors and profunctors in any $\mathscr{V}$-enriched category $\mathscr{C}$. The paper culminates in the impressively general and elegant[3] result that the bicategory of $\mathscr{V}$-Mealy maps is simply the Kleisli bicategory of the lax idempotent 2-monad of $\mathscr{V}$-copower completion.[4]

In a joint work [16] we explore how KSW's 'circuits' and Guitart's Mac connect via a *local adjunction* [67,73], and can be used to enhance categorical automata into

[1] Interestingly enough, KSW category can be seen as a lax analogue of the category of 'categories with endofunctor' upon which one builds the *Spanier-Whitehead stabilization* of the category of (pointed) CW-complexes, a staple construction in stable homotopy theory [123],[87, Chapter I].

[2] Note in passing that this is related to Betti, Kasangian, and Rosebrugh idea as two-sided fibrations and profunctors are well-known equivalent ways to present the same bicategory.

[3] The reader suspecting that this is an overstatement shall rest with the thought that this straightforward statement bestows the bicategory $\mathscr{V}$-Mly with a clear-cut universal property generalizing, in one fell swoop, KSW and Guitart's approach to every base of enrichment.

[4] The reader will have noticed a repeating theme: categories that naturally arise organizing computational machines share a universal property of Kleisli type (they are initial in some sense, for ways of factoring a certain monad), and the monad is 'of property type', i.e. it is a 2-monad of cocompletion under certain shapes [80,136].

widgets 'typed' over a bicategory with possibly more than one object; in short, it allows the passage from a bicategory of automata to *automata in a bicategory*, drawing some ideas from Bainbridge's [6,7]. Despite its relative obscurity, likely due to its cutting-edge nature, Bainbridge's work recognized the importance of bicategory theory as a foundational language for the theory of abstract automata and, in particular, proposed the idea of left/right Kan extensions along an 'input scheme' to analyze behaviour and minimization.

To sum up, we find ourselves in the following situation today: a forgotten school of category theorists hid an exciting claim behind a curtain of 2-dimensional algebra:

> A piece of *formal category theory* as envisioned by [47,49,52,120,127,128,131] serves as the mathematical foundation of abstract state machines.

This intriguing hypothesis is scattered across various sources, often unaware of each other; it has been hinted at multiple times and continues to leave traces of its presence for those willing to follow it. We are left with a conjecture and a clear work plan: can this fundamental guiding principle be taken seriously and formalized? Whoever is willing to take up the challenge of verifying this claim is now tasked with lifting the curtain and exploring a rich fauna of categorical widgets.

The present work grafts on top of the wide branches of this overarching project, studying categorical automata theory specialized to the *differential 2-rig* (a notion introduced by the author in [86]) of Joyal's combinatorial species. The category Spc of species is a presheaf topos equipped with a plethora of tightly-knit monoidal structures interacting with a differential structure; this richness implies that when used as an ambient category for monoidal/functorial automata, it gives rise to an interesting theory that, when stated at the correct level of abstraction, is 'stable under small perturbations', which means that similar results to the ones presented here export without much effort to presheaf categories equipped with a plethystic substitution operation, such as *coloured* species [99], *linear* species (both in the sense of [82] and in the sense of k-Mod-enriched, [4,36]), *Möbius* species [96], nominal sets [108],... and it allows to predict what happens when abstract automata are interpreted in a differential 2-rig other than Spc, generalizing Theorem 4.

1.1 Outline of the paper

The basic terminology about the category of species that we need is classical, drawing upon various sources such as [11,32,135,132]; we rework an equally 'classical' construction of the categories $\mathsf{Mly}_{\mathcal{K}}(F, B)$ and $\mathsf{Mre}_{\mathcal{K}}(F, B)$, drawing from [25,51]. In Proposition 5, we introduce the concept of 'ω-differential limit', as an intuition for what the terminal object in $\mathsf{Mly}_{\mathcal{K}}(F, B)/\mathsf{Mre}_{\mathcal{K}}(F, B)$ should represent; the terminology is somewhat borrowed from ergodic theory (specifically, the notion of ω-limit, see [40, Def. 1.12]). Later, in Section 3.1, we thoroughly explore the fibrational properties of the $\mathsf{Mly}_{\mathcal{K}}$ construction, yielding the 2-fibration of the *total Mealy 2-category* **Mly**, along with two-sided fibrations [119] $\mathcal{Mly}_{\mathcal{K}}/\mathcal{Mre}_{\mathcal{K}}$ allowing to consider all dynamics and all outputs at the same time, coherently. In (13) we define the *monoidal Mealy fibration* as a particular instance of this construction. The fundamental result of [75], defining the KSW category of a monoidal category $(\mathcal{K}, \otimes)$ arises (Theorem 3) when the profunctor associated to the monoidal Mealy two-sided fibration carries the structure of a promonad, of which $\mathrm{KSW}(\mathcal{K}, \otimes)$ is the Kleisli object. In Proposition 7 we address the issue of lifting accessibility from $\mathcal{K}$ to $\mathsf{Mly}_{\mathcal{K}}/\mathsf{Mre}_{\mathcal{K}}$, consolidating the idea that nice properties of the ambient category lift easily to its category of automata. Interestingly, assuming $\mathcal{K}$ is a differential 2-rig in the sense of [86], $\mathscr{Mly}_{\mathcal{K}}$ and $\mathscr{Mre}_{\mathcal{K}}$ are differential 2-rigs: an upshot of [86] is that differential structures are 'difficult to create', and yet categories of $\mathcal{K}$-valued automata exhibit an additional differential 2-rig structure, simply but not trivially related to $\mathcal{K}$.

Finally we turn to the task of studying (Mealy) automata in species, focusing on the particular case where $\mathcal{K}$ is the category of Section 2; given its structure of differential 2-rig, we are particularly interested in studying *differential dynamics*, i.e. in studying categories $\mathsf{Mly}_{\mathsf{Spc}}(F, B)$ where the generator F of dynamics is induced by the derivative functor. Given the results in [110], recalled in Theorem 1, there is plenty of choice for such F's: the triple of adjoints $L \dashv \partial \dashv R$ generates four functors, a comonad-monad adjunction $L\partial \dashv R\partial$ and a monad-comonad adjunction $\partial L \dashv \partial R$ (paying tribute to the 'twelvefold way' of [116], we dub the study of this quadruple of pairwise adjoint functors the 'fourfold way'); each of these adjunctions generate monads or comonads $R\partial L\partial$, $L\partial R\partial$,$\partial R\partial L$, $\partial L\partial R$ and all these are finitely accessible functors because R is.

2 The category of species

Issues of page count force us to condense a wealth of material in a small space; the reader will find excellent introductory texts and surveys on the category of species in [11,32,135,132] or in the original papers by Joyal [72,71].

After defining the category Spc of combinatorial species, we recall its various monoidal structures and outline how they relate, with particular attention to the *Day convolution* monoidal structure, and its *differential 2-rig* structure in the sense of [86], with particular attention to the fact that the derivative functor $\partial : \mathsf{Spc} \to \mathsf{Spc}$ has both a left and a right adjoint. We study co/monoids in $(\mathsf{Spc}, \otimes_{\mathrm{Day}})$, since monoidal automata theory in a category with countable sums forces us to understand the structure of the subcategory of $\otimes_{\mathrm{Day}}$-monoids at a fundamental level.

Definition 1 (Species and $\mathcal{V}$-species). *Let S be a set, and $\mathcal{V}$ a symmetric monoidal closed category admitting all limits and colimits. The category $(S, \mathcal{V})$-Spc of* (S-colored) $\mathcal{V}$-species *is defined as the category of functors $\mathfrak{F} : \mathsf{P}[S] \to \mathcal{V}$, where $\mathsf{P}[S]$ is the free symmetric monoidal category on S, regarded as a discrete category.*

We will particularly be interested in the category $(1, \mathsf{Set})$-Spc (that we dub simply Spc in the following), where 1 is a singleton.[5] The category $\mathsf{P} = \mathsf{P}[1]$ is called the *groupoid of natural numbers*, having as objects the nonnegative integers $[0], [1], [2], \dots$ and where morphisms $n \to m$ are the symmetric group S_n of permutations of the set $[n] = \{1, \dots, n\}$ (so in particular $[0]$ is the empty set and S_0 is the trivial group) if

[5] Other possible choices for $\mathcal{V}$ are the category Mod_R of modules over a ring R (if R is a field, we call a Mod_k-species just a k-linear species, see [84] for a comprehensive introduction) or the category Top_* of pointed topological spaces equipped with the smash product [121, 3.6.2] (for applications to algebraic topology, see e.g. [91]; for a broader notion of operad cf. the excellent readings [22,124]).

$n = m$, and the empty set otherwise. As such, P is the skeleton of the *groupoid of finite sets* Bij, the category having objects the finite sets $A, B, \dots$ and morphisms $A \to B$ the set of all bijections between A and B (so, in particular, if A and B do not have the same cardinality, $\mathsf{Bij}(A, B)$ is empty). Note that Bij is the core (=larger subcategory that is a groupoid) of the category of finite sets. The (commutative, i.e. strictly symmetric) monoidal structure on P is given by sum of natural numbers, i.e. $[n] \oplus [m] = [n+m]$, the unit is [o] and permutations act by juxtaposition. In the following we denote a species as $\mathfrak{F} : \mathsf{P} \to \mathsf{Set}$ and call an element $s \in \mathfrak{F}[n]$ a *species of* $\mathfrak{F}$*-structure.*

Corollary 1. *The universal property of* P *entails that there is an isomorphism of categories* $\mathsf{P} \cong \sum_n S_n$ *where the right-hand side is the coproduct in the category of groupoids and as a consequence* $\mathsf{Spc} \cong \prod_n \mathsf{Set}^{S_n}$ *where each* Set^{S_n} *is the category of left* S_n*-set.*

As a consequence, species can equivalently be presented as a symmetric sequence $\{X_n \mid n \geq 0\}$ *of sets, each of which is equipped with a (left)* S_n*-action* $S_n \times X_n \to X_n$*.*

Definition 2 (Change of base for species). *Let* $\mathscr{V}$ *be a monoidal category monadic over* Set *such that the functor* $K : \mathscr{V} \to \mathsf{Set}$ *is lax monoidal (for example the forgetful functor* $U : \mathsf{Mod}_R \to \mathsf{Set}$*); then there is a* base change *adjunction* $F_* : \mathsf{Spc} \rightleftarrows \mathscr{V}\text{-}\mathsf{Spc} : U_*$ *induced through the free-forgetful* $F \dashv U$*. For example, if* $F : \mathsf{Set} \to \mathsf{Mod}_k$ *is the free* k*-vector space functor, we denote* $k\langle L\rangle$ *the* k*-linear species* $F_* L$ *induced by a* Set*-species* L*.*

Example 1 (Some important species). Many more examples of species can be found in [11, Ch. 1].

- ES1) Given an object V of $\mathscr{V}$, there is a unique symmetric monoidal $\mathscr{V}$-species c_V sending $[n]$ to $V^{\otimes n}$. If $V = I$ is the monoidal unit, c_I is called the 'exponential species' $\mathfrak{E}$. The exponential Set-species is just the constant functor at the terminal object.[6]
- ES2) The species $\wp$ of subsets sends an n-set A to the 2^n-set of all its subsets; a permutation acts in an obvious way, since a bijection $\sigma : A \to A$ induces a bijection $\sigma^* : 2^A \to 2^A$ by functoriality.
- ES3) The species $\mathfrak{L}$ of total orders[7] sends $[n]$ to the set of total orders on $[n]$, identified with the *set* $|S_n|$ of bijections of $[n]$, over which S_n acts by left multiplication.
- ES4) The species $\mathfrak{S}$ of permutations sends each finite set $[n]$ into the (carrier of the) symmetric group on n letters, S_n. The symmetric group acts on itself by conjugation: if $\tau \in S_n$, $\sigma : S_n \to S_n$ is the map sending $\tau \mapsto \sigma\tau\sigma^{-1}$.
- ES5) The species $\mathfrak{Cyc}$ of *oriented cycles* sends a finite set $[n]$ to the set of inequivalent (i.e. not related by a cyclic permutation) ways to sit n people at a round table, or more formally, in the set of *cylic orderings* of $\{x_1, \dots, x_n\}$. As $\mathfrak{Cyc}[n]$ identifies with the set of cosets S_n/C_n (C_n the cyclic group), one derived that $|\mathfrak{Cyc}[n]| = (n-1)!$.

[6] In a serendipitous choice, the notation $\mathfrak{E}$ for this species hints at the same time that $\mathfrak{E}$ is the species of sets, or *éspèce des ensembles*, and that it's an analogue of the exponential function, as $\partial\mathfrak{E}[n] = \mathfrak{E}[n]$, for the derivative functor of Remark 4.

[7] Elsewhere customarily called the species of *linear* orders, but this might conflict with linear as in 'k-linear' if k is a ring.

The category of species exhibits a fairly rich structure that we now review.[8]

Proposition 1. Spc *is the free cocompletion under small colimits [125, Remark 2.29] of* P*; as such, for every cocomplete category* $\mathcal{D}$ *there is an equivalence of categories*

$$\mathsf{Cat}(\mathsf{P},\mathcal{D}) \cong \{\textit{colimit preserving functors } \mathsf{Spc} \to \mathcal{D}\} \tag{1}$$

given by 'Yoneda extension' [85, Ch. 2].

Proposition 2. *Following [2,57,71]* Spc *is the (nonfull) subcategory of* analytic endofunctors *of* Set*, i.e. those endofunctors* $F : \mathsf{Set} \to \mathsf{Set}$ *such that, if* $J : \mathsf{Bij} \to \mathsf{Set}$ *is the tautological functor* $[n] \mapsto [n]$*, the left Kan extension of* FJ *along* J *coincides with* J*. The usual coend formula [88] to express* $\mathrm{Lan}_J FJ$ *entails that* F *is analytic if and only if it acts on a set* X *as*

$$FX \cong \int^n F[n] \times X^n \tag{2}$$

i.e. if and only if F *admits a 'Taylor expansion'* $\sum_{n=0}^{\infty} F[n]\frac{X^n}{n!}$*; hence the name. The series* $g_F(X) = \sum_{n=0}^{\infty} |F[n]|\frac{X^n}{n!} \in \mathbb{Q}[\![X]\!]$*, where* $|S|$ *denotes the cardinality of a set, is called the (exponential)* generating series *[11, §1.2] of the species* F*.*

Proposition 3 ([86, §5]). Spc *is the free cocomplete 2-rig on a singleton; as such, given a cocomplete 2-rig* $\mathcal{R}$ *there is an equivalence of categories*

$$\mathcal{R} \cong \{\textit{colimit preserving 2-rig functors } \mathsf{Spc} \to \mathcal{R}\} \tag{3}$$

In [86, §5] we observe how to construct the free cocomplete (symmetric) 2-rig on a given category $\mathcal{A}$ *it suffices to take the free (symmetric) monoidal category on* $\mathcal{A}$*, call it* $\mathsf{P}[\mathcal{A}]$ *and subsequently, its free cocompletion* $\mathsf{Cat}(\mathsf{P}[\mathcal{A}]^{\mathrm{op}},\mathsf{Set})$*. The notion of morphism of 2-rigs is given indirectly as pseudomorphism for the particular 'doctrine of* $\boldsymbol{D}$*-rigs' in study.*

This last characterization requires a more fine-grained analysis of the various monoidal structures Spc can be equipped with.

Remark 1. The category Spc of species carries

MS1) the *Cartesian* (or *Hadamard* [4, 8.1.2]) monoidal structure, the product of species being taken pointwise; the monoidal unit for the Hadamard product is the species that is constant at the singleton. Dually, the *coCartesian* monoidal structure, the coproduct of species being taken pointwise (together with the structure above, Spc is $\times$-distributive and forms a 'biCartesian closed' category in the sense of [122]); however, its biCartesian structure is not very interesting, compared to

[8] An important additional universal property we do not need in our analysis is that Spc is a Grothendieck topos, precisely the classifying topos [89, Ch. VIII] for $\mathcal{P}$-torsors, where $\mathcal{P}$ is the category P regarded as a groupoid.

MS2) the *Day convolution* (or *Cauchy* [4, 8.1.2]) monoidal structure, given by the universal property of Spc as the free monoidally cocomplete category on P [62] as the coend

$$(F \otimes_{\mathrm{Day}} G)[p] := \int^{mn} F[m] \times G[n] \times \mathsf{P}(m+n, p) \tag{4}$$

(Note in passing that the $\otimes_{\mathrm{Day}}$-monoidal structure is symmetric and closed with an internal hom $\{-,-\}_{\mathrm{Day}}$.) In particular, P is monoidally equivalent to the subcategory of Spc spanned by representables, and thus the $\otimes_{\mathrm{Day}}$-monoidal unit is $y[0]$.

MS3) the *substitution* (or *plethystic*, cf. [95,103]) monoidal structure, defined for $F, G : \mathsf{P} \to \mathsf{Set}$ as $(F \circ G)[p] = \int^n Fk \times G^{\otimes_{\mathrm{Day}} k}[p]$, where $G^{\otimes_{\mathrm{Day}} k} := G \otimes_{\mathrm{Day}} G \otimes_{\mathrm{Day}} \cdots \otimes_{\mathrm{Day}} G$ (k times). The $\circ$-monoidal unit is the representable $y[1]$. Note in passing that the $\circ$-monoidal structure is *not* symmetric, and only *right* closed, i.e. only $\_ \circ G$ has a right adjoint.

All these monoidal structures are tightly related:

Remark 2. The Hadamard and Day convolution product give Spc the structure of a *duoidal category* in the sense of [34]: $(\mathsf{Spc}, \times, \otimes_{\mathrm{Day}})$ and $(\mathsf{Spc}, \otimes_{\mathrm{Day}}, \times)$ [4, 8.13.5] are both duoidal; *positive* species, i.e. those for which $F[\varnothing] = \varnothing$ form a duoidal category under substitution and Hadamard product, [B.6.1, *ibi*]. All these results extend to $\mathscr{V}$-species. The plethystic structure makes Spc monoidally equivalent to the category of analytic functors under composition [2,71].

Remark 3. As an additional demonstration of how tightly the Hadamard, Cauchy and plethystic structures are related, observe how all these identification between combinatorial species hold [11]:

CI1) the species of subsets $\wp : A \mapsto 2^A$ is isomorphic to $\mathfrak{E} \otimes_{\mathrm{Day}} \mathfrak{E}$;
CI2) the species $\mathfrak{S}$ of permutations of ES4 is isomorphic to the substitution $\mathfrak{E} \circ \mathfrak{Cyc}$;
CI3) more generally, for every species $\mathfrak{F}$ the substitution $\mathfrak{E} \circ \mathfrak{F}$ sends A to a r-partition $(U_1, \dots, U_r)$ of A and picks a $\mathfrak{F}$-structure on each U_i.

An important structure enjoyed by Spc that we will analyze in this paper is that of a *differential 2-rig*: the notion was introduced in [86] as a unifying language for instances of a monoidal category $(\mathscr{C}, \otimes, I)$ where each $A \otimes \_, \_ \otimes B$ is cocontinuous and an endofunctor ∂ is 'linear and Leibniz' in the following sense.

As it is true in all presheaf categories, the tensor product $\otimes_{\mathrm{Day}}$ preserves colimits separately in each variable (i.e., each $A \otimes_{\mathrm{Day}} \_$ and $\_ \otimes_{\mathrm{Day}} B$ is cocontinuous); moreover, the following is true.

Remark 4 (The differential structure of Spc*).* The category Spc of species is equipped with a 'derivative' endofunctor $\partial : \mathsf{Spc} \to \mathsf{Spc}$ (cf. [11, §1.4 and *passim*]; ∂F is the species sending $[n]$ to $F[n \oplus [1]]$) such that

D1) ∂ is 'linear', i.e. it preserves all colimits (in particular, coproducts);
D2) ∂ is 'Leibniz', i.e. it is equipped with tensorial strengths $\tau' : \partial A \otimes B \to \partial(A \otimes B)$ and $\tau'' : A \otimes \partial B \to \partial(A \otimes B)$ such that the unique map induced by τ', τ'' from the coproduct of their domains is invertible, to the effect that ∂ 'satisfies the Leibniz rule'

$$\partial A \otimes B + A \otimes \partial B \cong \partial(A \otimes B). \tag{5}$$

Definition 3. *Every monoidal category* $(\mathcal{K}, \otimes)$ *equipped with an endofunctor* ∂ *that satisfies the same three properties is called a* differential 2-rig *(for the doctrine of colimits) in [86].*

In the case of species, the proof that $\partial(F \otimes_{\text{Day}} G) \cong \partial F \otimes_{\text{Day}} G + F \otimes_{\text{Day}} \partial G$ appears in Joyal's original papers introducing combinatorial species. Moreover, it was known to Joyal that ∂ satisfies the 'chain rule' in that $\partial(F \circ G) \cong (\partial F \circ G) \otimes_{\text{Day}} \partial G$; cf. [86, Theorem 5.18] for a conceptual proof of this latter result. To a very large extent, the combinatorial differential calculus of species agrees with the classical differential calculus of formal power series. In particular, observe that $g_{\partial F}(X)$ is the formal derivative $\frac{d}{dX} g_F(X)$ of the series in Proposition 2.

Remark 5. Part of the fairly rich structure enjoyed by the differential 2-rig $(\mathsf{Spc}, \otimes, \partial)$ can be explained with the fact that ∂ also preserves all *limits*: ∂F is precomposition with the $\_ \oplus [1]$ functor; but then, call $\Delta = \_ \oplus [1]$, the left (resp., right) adjoint to ∂ is the left (resp., right) Kan extension along Δ, which exists since Spc is a presheaf category.[9]

We just proved the following result:

Theorem 1. *The derivative functor* $\partial : \mathsf{Spc} \to \mathsf{Spc}$ *sits in a triple of adjoints* $L \dashv \partial \dashv R$*, and* L, R *are obtained as Kan extensions.*

This fact was first observed in [110], where the explicit descriptions

$$LF : A \mapsto \sum_{a \in A} F[A \setminus \{a\}] \qquad RF : A \mapsto \prod_{a \in A} F[A \setminus \{a\}] \tag{6}$$

are given in terms of F as a functor $\mathsf{Bij} \to \mathsf{Set}$, and some useful combinatorial identities expressing $L\partial, R\partial, \partial L, \partial R$ in simpler terms are also analyzed.

Notation 2 (Scopic 2-rig). *We introduce the terminology* scopic[10] *2-rig to refer to a differential 2-rig* $(\mathcal{R}, \otimes, D)$ *whose derivative functor* D *has both a left and a right adjoint.*

Algebraic structures and co/algebras in Spc. We end the section reviewing the characterization of monoids, comonoids and Hopf monoids in Spc. First of all, Hadamard co/monoids are simply co/monoid-valued species, i.e. functors $F : \mathsf{P} \to \mathsf{Mon}$ or $\mathsf{P} \to \mathsf{Comon}$ into the categories of monoids and comonoids in Set (and this result extends to $\mathcal{V}$-species, a Hadamard monoid in $\mathcal{V}$-Spc being just a functor $\mathsf{P} \to \mathsf{Mon}(\mathcal{V})$).

Cauchy co/monoids (i.e. co/monoids for the Day convolution, whence our preference for calling them Day co/monoids) are far more interesting, as well as substitution co/monoids (the latter are called *co/operads* and have an extremely long history,

[9] An alternative proof of the same fact is in terms of the Day convolution structure: one sees that there is a natural isomorphism $\partial F \cong \{y[1], F\}_{\text{Day}}$ where $\{-,-\}_{\text{Day}}$ is the internal hom, and $y[1] = \mathsf{P}(1, -)$ the corepresentable functor on $[1]$; now, certainly $\{y[1], -\}_{\text{Day}}$ must have $y[1] \otimes_{\text{Day}} -$ as left adjoint, but since in every presheaf category representables are tiny objects, ∂ must also be cocontinuous, hence a left adjoint by the special adjoint functor theorem.

[10] From the Proto-Indo-European root **speḱ-*, derived the Latin word *speciēs* and the Greek verb σκοπέω, related to the verb 'to see'.

excellent surveys geared towards the different areas of Mathematics using them are [22,32,77,91]). The first remark on $\otimes_{\text{Day}}$-co/monoids is simply that there aren't any among representables.

Remark 6. There are no nontrivial representable $\otimes_{\text{Day}}$-magmas, for the simple reason that the subcategory spanned by representables is monoidally equivalent to $(\mathsf{P}, \oplus)$, and in the latter a binary operation $[n] \oplus [n] = [2n] \to [n]$ can exist only if $2n = n$. For a similar reason, there are no nontrivial k-ary cooperations $[n] \to [n]^{\oplus k}$.

Remark 7. It is worth to explicitly spell out what a $\otimes_{\text{Day}}$-monoid (M, μ, η) in Spc must be made of:

- the unit consists of a species morphism $\eta : y[0] \to M$ which by Yoneda is just an element $e \in M[0]$.
- the multiplication splits into a cowedge $\mu_{pq} : M[p] \times M[q] \to M[n]$ for each pair of integers p, q such that $p + q = n$, natural for the action of symmetric groups, under the shuffling maps $S_p \times S_q \to S_{p+q}$ sending a pair of permutations (σ, τ) to the one acting as σ on $\{1, \dots, p\}$ and as τ on $\{p + 1, \dots, p + q\}$.

The following is implied joining [2, Example 2.3] and adapting [4, 8.16]: in particular, the species $\mathfrak{L}$ of Example 1 has a convenient universal property.

Proposition 4. *([11, p. 7], [4, §8.1]) The species $\mathfrak{L}$ of total orders is the free monoid on $y[1]$. The species $\mathfrak{L}_+$ of* nonempty *linear orders is the free semigroup on $y[1]$. Thus,*

$$\mathfrak{L} \cong \textstyle\sum_{n\geq 0} y[n] \qquad\qquad \mathfrak{L}_+ \cong \textstyle\sum_{n\geq 1} y[n]. \tag{7}$$

(A terminological note. [4] calls 'positive' what we tend to dub 'nonempty', considering species as monadic over graded vector spaces.) In fact, in a k-linear setting (k a field) the structure of $\mathfrak{L}$ is way richer: $k\langle\mathfrak{L}\rangle$ (cf. Definition 2; it's the species assigning to $[n]$ the k-vector space having the set $\mathfrak{L}[n]$ as a basis) carries the structure of a *Hopf monoid*. Following Remark 7, the monoid structure of $\mathfrak{L}$ arises as a cowedge $\mathfrak{L}[p] \times \mathfrak{L}[q] \to \mathfrak{L}[n]$ for every $p + q = n$, defined as $(l, l') \mapsto l \cdot l'$ where the later is the *ordinal sum* or concatenation of the linear orders l on $[p]$ and l' on $[q]$; ordinal sum is an associative operation, equivariant under the shuffling maps. The unit is the only element of $\mathfrak{L}[1]$.

The Hopf monoid structure of $k\langle\mathfrak{L}\rangle$ is extensively studied and described in [4, §8.5].

Co/algebras for endofunctors of Spc This subsection studies algebras and coalgebras for a few interesting endofunctors M defined over Spc. Despite its naturality, this idea is seemingly unexplored thus far.

It becomes particularly intriguing to explore the interactions between M and the structures on Spc mentioned in Remark 1, Remark 4; clearly, this is essential to study (M, B)-automata, defined in Definition 5 as a pullback along M-algebras.

Definition 4 (The category $\mathsf{Spc}^{\mathfrak{L}}$). *The category $\mathsf{Spc}^{\mathfrak{L}}$ is, up to equivalence, described as any of the following:*

L1) *the category of endofunctor algebras for $y[1] \otimes_{\text{Day}} -$;*
L2) *the category of endofunctor coalgebras for ∂;*

L3) *the Eilenberg–Moore category of the monad* $\mathfrak{L} \otimes_{\text{Day}} \_$*;*
L4) *the coEilenberg–Moore category of the comonad* $\{\mathfrak{L}, -\}_{\text{Day}}$.

These identifications follow from the freeness of $\mathfrak{L}$ and the general fact that whenever $F \dashv G$ is an adjunction between endofunctors, $\mathsf{Alg}(F) \cong \mathsf{coAlg}(G)$.

Representing objects of $\mathsf{Spc}^{\mathfrak{L}}$ as Eilenberg–Moore algebras is particularly convenient, as a $\mathfrak{L}$-module is the same thing as a $\otimes_{\text{Day}}$-monoid homomorphism $\mathfrak{L} \to \{F, F\}_{\text{Day}}$, which since $\mathfrak{L}$ is the free monoid generated on $y[1]$, amounts to a single element of $\{F, F\}_{\text{Day}}[1]$; equivalently, if one uses characterization L1 above, a structure of type $y[1] \otimes_{\text{Day}}$ on $[n]$ consists of a choice of point in $[n]$, together with an F-structure on the complement of that point.[11]

Remark 8. Limits and colimits in $\mathsf{Spc}^{\mathfrak{L}}$ are computed exactly as in Spc, i.e. pointwise (since Spc is monadic over $\mathsf{Set}^{\mathbb{N}} = \prod_{n \geq 1} \mathsf{Set}$), given that $\mathsf{Spc}^{\mathfrak{L}}$ is at the same time a category of algebras (for $\mathfrak{L} \otimes_{\text{Day}} \_$, hence limits are created in Spc) and of coalgebras (for the right adjoint comonad $\{\mathfrak{L}, -\}_{\text{Day}}$, hence colimits are created in Spc). We just proved that

Lemma 1. *The terminal object of* $\mathsf{Spc}^{\mathfrak{L}}$ *is the exponential species of Example 1, whence the isomorphism* $\partial\mathfrak{E} \cong \mathfrak{E}$ *characterizing* $\mathfrak{E}$ *as a 'Napier object' of the differential 2-rig of species.*[12]

Armed with these explicit computations, we can attempt to unveil the structure of the category $\mathsf{Spc}^{\mathfrak{L}}$ in any of the equivalent forms given in Definition 4 as a building block of $\mathsf{Mly}_{\mathsf{Spc}}(\mathfrak{L}, \_)$.

We now collect some examples of: a species that has only a few structures of $\mathfrak{L}$-algebra (=structures of ∂-coalgebra); a species that has at least uncountably many; a species with *no* such structure as a Set-species, that however becomes interesting when 'changing base' (cf. Definition 2).

Example 2. Structures of ∂-coalgebra on the species of subsets of ES2 correspond to S_n-equivariant maps $\theta : \wp \to \partial\wp$ and using the Leibniz rule over the isomorphism $\wp \cong E \otimes_{\text{Day}} E$ of [11, §1.3, Eq. (33)] one gets that $\theta : \wp \to \wp + \wp$. Using elementary group theory on the components θ_A one sees that there are only four such θ: embedding a subset $U \subseteq A$ in the first summand, embedding a subset $U \subseteq A$ in the second summand, embedding $U^c = A \smallsetminus U$ in the first summand, embedding $U^c = A \smallsetminus U$ in the second summand.

Example 3. [11, Example 9, (37)] yields $\partial\mathfrak{L} \cong \mathfrak{L} \otimes_{\text{Day}} \mathfrak{L}$, whence a natural choice for a coalgebra structure $s : \mathfrak{L} \to \partial\mathfrak{L}$, given a finite set A, is specified on components s_A in terms of a choice of decomposition $A = I \sqcup J$ and a splitting of the total order on A as a total order on I and a total order on J. This choice is made independently for every finite set A, so this argument shows that there is an uncountable infinity of coalgebra structures on $\mathfrak{L}$.

[11] One can read off the fact that these descriptions are equivalent from the end defining $\{F, F\}_{\text{Day}}[n]$, cf. [77, Equation (2.6)].

[12] The rationale behind the terminology is that, evidently, 'exponential object' already has a different, conflicting meaning.

Example 4. Let $\mathfrak{Cyc}$ be the species of cyclic orders, Example 1.ES5; then, we immediately get $\partial\mathfrak{Cyc} \cong \mathfrak{L}$ from manipulating generating series. A ∂-coalgebra structure on $\mathfrak{Cyc}$ now would be a natural transformation $\vartheta : \mathfrak{Cyc} \to \mathfrak{L}$, and no such map can exist by cardinality reasons: since $\mathfrak{Cyc}[n]$ identifies with the coset space $S_n/\mathbb{Z}_n$, over which S_n acts transitively, an S_n-equivariant map $\vartheta_n : \mathfrak{Cyc}[n] \to S_n$ must be surjective (the translation action $S_n \times \mathfrak{Cyc}[n] \to \mathfrak{Cyc}[n] : (\sigma, \tau) \mapsto \sigma\tau$ is also transitive). Yet, $|S_n| = n! > (n-1)! = |\mathfrak{Cyc}[n]|$.

Example 5. Let $\mathfrak{S}$ be the species of permutations of Example 1.ES4; from Remark 3 it follows that $\partial\mathfrak{S} \cong \mathfrak{S} \otimes_{\mathrm{Day}} \mathfrak{L}$, so that ∂-coalgebra structures (i.e. Eilenberg–Moore algebras for $\mathfrak{L} \otimes_{\mathrm{Day}} \_$) correspond under adjunction to monoid homomorphisms $\mathfrak{L} \to \{\mathfrak{S}, \mathfrak{S}\}_{\mathrm{Day}}$.

3 Abstract automata in Spc

Let $\mathscr{K}$ be a category and $F : \mathscr{K} \to \mathscr{K}$ be an endofunctor that we think of as a categorification of a dynamical system and its iterates $F, F^2, F^3, \dots, F^n : \mathscr{K} \to \mathscr{K}$, cf. [3]. We also fix an object $B \in \mathscr{K}$ (an 'output' object, cf. [25,51]).

Definition 5. *We define the category* $\mathsf{Mly}_{\mathscr{K}}(F, B)$ *and* $\mathsf{Mre}_{\mathscr{K}}(F, B)$ *as the following strict 2-pullbacks in* Cat *respectively:*

$$\begin{array}{ccc} \mathsf{Mly}_{\mathscr{K}}(F,B) & \longrightarrow & F/B \\ \downarrow & \lrcorner & \downarrow U \\ \mathsf{Alg}(F) & \xrightarrow[V]{} & \mathscr{K} \end{array} \qquad \begin{array}{ccc} \mathsf{Mre}_{\mathscr{K}}(F,B) & \longrightarrow & \mathscr{K}/B \\ \downarrow & \lrcorner & \downarrow U' \\ \mathsf{Alg}(F) & \xrightarrow[V]{} & \mathscr{K} \end{array} \tag{8}$$

where $\mathsf{Alg}(F)$ *is the category of* endofunctor algebras *of* F*,* F/B *the comma category of arrows* $FX \to B$*, and* $\mathscr{K}/B$ *the comma category of arrows* $X \to B$ *(and* U, V, U', V' *are the most obvious forgetful functors).*

Remark 9 (Limits and colimits in categories of automata). If F admits a right adjoint R, and $\mathscr{K}$ is complete and cocomplete, so are $\mathsf{Mly}_{\mathscr{K}}(F, B)$ and $\mathsf{Mre}_{\mathscr{K}}(F, B)$; this can be easily argued using an argument in [88, V.6, Ex. 3] and the fact that U, U' create colimits and connected limits, together with the fact that $F/B \cong \mathscr{K}/RB$; then, the terminal object of $\mathsf{Mly}_{\mathscr{K}}(F, B)$ is $\prod_{n\geq 1} R^n B$ and the terminal object of $\mathsf{Mre}_{\mathscr{K}}(F, B)$ is $\prod_{n\geq 0} R^n B$.

Remark 10 (Accessibility of categories of automata). Repeatedly applying the completeness theorem of the 2-category Acc of accessible categories [90, Ch. 5] one can prove that if $\mathscr{K}$ is locally presentable (say for a regular cardinal κ) and F is κ-accessible, then $\mathsf{Mly}_{\mathscr{K}}(F, B), \mathsf{Mre}_{\mathscr{K}}(F, B)$ are both locally presentable (but in general, for a much higher cardinal κ).

Remark 11. A particular instance of Remark 9 is when $\mathscr{K}$ is monoidal and $F : \mathscr{K} \to \mathscr{K}$ is the tensor product $A \otimes -$ for a fixed object of $\mathscr{K}$. Then, we shorten $\mathsf{Mly}_{\mathscr{K}}(F, B)$ and $\mathsf{Mre}_{\mathscr{K}}(F, B)$ to $\mathsf{Mly}_{\mathscr{K}}(A, B)$ and $\mathsf{Mre}_{\mathscr{K}}(A, B)$ and we observe that

- if $\mathcal{K}$ has countable sums, $\mathsf{Alg}(F) = \mathsf{Alg}(A \otimes -)$ is the Eilenberg-Moore category of the monad $A^* \otimes -$ where $A^* := \sum_{n=0}^{\infty} A^{\otimes n}$ is the free monoid on A;
- if $\mathcal{K}$ is monoidal closed, complete and cocomplete, then $\mathsf{Mly}_{\mathcal{K}}(A, B)$ and $\mathsf{Mre}_{\mathcal{K}}(A, B)$ are complete and cocomplete; if $\mathcal{K}$ is locally κ-presentable, so are $\mathsf{Mly}_{\mathcal{K}}(A, B)$ and $\mathsf{Mre}_{\mathcal{K}}(A, B)$ (generally, for a larger cardinal $\kappa' \gg \kappa$). The terminal object in $\mathsf{Mly}_{\mathcal{K}}(A, B)$ is $[A^+, B]$, A^+ being the free semigroup on A (resp., in $\mathsf{Mre}_{\mathcal{K}}(A, B)$ it's $[A^*, B]$, A^* being the free monoid).

Unwinding Definition 5 in this particular case, the typical object $\left\|\frac{E}{d,s}\right\|$ of $\mathsf{Mly}_{\mathcal{K}}(A, B)$ is a span as in the left of the following diagram, and the typical object $\left\|\frac{E}{d,s}\right|$ of $\mathsf{Mre}_{\mathcal{K}}(A, B)$ a (disconnected) diagram as in the right

$$\left\|\frac{E}{d,s}\right\|: E \xleftarrow{d} A \otimes E \xrightarrow{s} B \qquad \left\|\frac{E}{d,s}\right|: E \xleftarrow{d} A \otimes E, E \xrightarrow{s} B. \tag{9}$$

Remark 12. Remarks 9, 10, 11 all apply to $\mathcal{K} = \mathsf{Spc}$ considered with the Day convolution structure (and in fact to all $\mathcal{V}$-Spc when $\mathcal{V}$ is complete, cocomplete and monoidal closed). In particular, for every fixed combinatorial species $B : \mathsf{P} \to \mathsf{Set}$ we can easily study $\mathsf{Mly}_{\mathsf{Spc}}(L, B) = \mathsf{Mly}_{\mathsf{Spc}}(y[1], B)$ as the category having objects the diagrams $E \xleftarrow{d} y[1] \otimes_{\text{Day}} E \xrightarrow{s} B$, or more concisely as the category obtained as the pullback $\mathsf{Spc}^{\mathfrak{L}} \times_{\mathsf{Spc}} (\mathsf{Spc}/B)$ where $\mathsf{Spc}^{\mathfrak{L}}$ is as in Definition 4.

Note that this is equivalent to the category of coalgebras for the functor $E \mapsto \partial B \times \partial E$. From this coalgebraic characterization, we deduce that

Proposition 5. *The terminal object of* $\mathsf{Mly}_{\mathsf{Spc}}(L, B)$ *is the 'ω-differential limit'*[13] *of* B *defined as*

$$\prod_{n \geq 1} \partial^n B \cong \prod_{n \geq 1} \{y[1]^{\otimes_{\text{Day}} n}, B\}_{\text{Day}} \cong \Big\{\sum_{n \geq 1} y[n], B\Big\}_{\text{Day}} = \{y[1]^+, B\}_{\text{Day}} \tag{10}$$

where again $y[1]^+$ *is the free semigroup on* $y[1]$*: given Proposition 4,* $y[1]^+ \cong \mathfrak{L}_+$.

Remark 13. Consider two endofunctors $F : \mathcal{K} \to \mathcal{K}$, $G : \mathcal{H} \to \mathcal{H}$. If $P : \mathcal{K} \to \mathcal{H}$ is a functor equipped with an intertwiner $\pi : GP \Rightarrow PF$ we can define a functor $\pi^* : \mathsf{Alg}(F) \to \mathsf{Alg}(G)$ by application of P and precomposition with π, a functor $\mathcal{K}/B \to \mathcal{H}/PB$ in the obvious way, and in turn a unique functor

$$\varpi^* : \mathsf{Mly}_{\mathcal{K}}(F, B) \longrightarrow \mathsf{Mly}_{\mathcal{H}}(G, PB) \tag{11}$$

[13] The name is chosen in analogy with the notion of ω-limit set of a dynamical system $f : X \to X$ defined over a metric space, see e.g. [40, Def. 1.12], where the (ω-)limit set of x under f is defined as

$$\omega(x, f) = \bigcap_{n \in \mathbb{N}} \overline{\{f^k(x) : k > n\}},$$

the topological closure of the 'eventual f-orbits' of x.

3.1 Fibrational properties of the Mly construction

The fact that Definition 5 is functorial in (F, B) motivates us to examine the fibrational properties of such associations $(F, B) \mapsto \mathsf{Mly}_{\mathcal{K}}(F, B)$ and $(F, B) \mapsto \mathsf{Mre}_{\mathcal{K}}(F, B)$. This yields total categories where all dynamics and all outputs can be considered simultaneously and coherently. The entire section takes place under the assumption that $\mathcal{K}$ is locally presentable.

Definition 6. *The* total Mealy 2-category **Mly** *is defined as follows:*

- *the objects are triples* $(\mathcal{K}; F, B)$ *where* $F : \mathcal{K} \to \mathcal{K}$ *is an endofunctor of a category* $\mathcal{K}$*, and* B *an object of* $\mathcal{K}$*;*
- *the morphisms* $(P, \pi, u) : (\mathcal{K}; F, B) \to (\mathcal{H}; G, B')$ *are triples where* $P : \mathcal{K} \to \mathcal{H}$ *is a functor,* $\pi : GP \Rightarrow PF$ *is an* intertwiner *natural transformation between* F *and* G *and* $u : PB \to B'$ *is a morphism;*
- *2-cells* $\gamma : (P, \pi, u) \Rightarrow (Q, \theta, v)$ *consist of natural transformations* $\gamma : P \Rightarrow Q$ *compatible with the intertwiners* π, θ *in the obvious sense, and such that* $v \circ \gamma_B = u$*.*

From such a domain **Mly***, sending* $(\mathcal{K}, F, B)$ *to* $\mathsf{Mly}_{\mathcal{K}}(F, B)$ *results in a strict 2-functor* **Mly** $\to$ **Cat** *(***Cat** *is the 2-category of categories, functors, natural transformations).*

It is, however, rarely needed to vary the domain $\mathcal{K}$ of the automata in study (but cf. Remark 18 for an instance of when this 'change of scalars' might be required). A simpler (=lower-dimensional) approach is convenient if we are content with keeping $\mathcal{K}$ fixed.

Definition 7 (The total categories of automata). *Definition 5 entails at once that the correspondence* $(F, B) \mapsto \mathsf{Mly}_{\mathcal{K}}(F, B)$ *is a (pseudo)functor of type* $\mathsf{Mly}_{\mathcal{K}} : \mathsf{Cat}(\mathcal{K}, \mathcal{K})^{\mathrm{op}} \times \mathcal{K} \to \mathsf{Cat}$*, i.e. a pseudo-profunctor* $\mathsf{Cat}(\mathcal{K}, \mathcal{K}) \looparrowright \mathcal{K}$ *from which we can extract a span*

$$\mathsf{Cat}(\mathcal{K}, \mathcal{K}) \xleftarrow{p} \mathscr{Mly}_{\mathcal{K}} \xrightarrow{q} \mathcal{K} \tag{12}$$

such that p *is a fibration,* q *is an opfibration,* p*-Cartesian lifts are* q*-vertical and* q*-opCartesian lifts are* p*-vertical, whose tip* $\mathscr{Mly}_{\mathcal{K}}$ *we call the* total Mealy category*.*

Similar considerations allow to construct the total Moore category $\mathscr{Mre}_{\mathcal{K}}$ *from the pseudo-profunctor* $(F, B) \mapsto \mathsf{Mre}_{\mathcal{K}}(F, B)$*, and obtain a two-sided fibration* $\mathsf{Cat}(\mathcal{K}, \mathcal{K}) \leftarrow \mathscr{Mre}_{\mathcal{K}} \to \mathcal{K}$*, the* total Moore category*.*

Remark 14. Unwinding the definition, it is easy to establish how reindexings of the total Mealy and Moore fibration act. In particular, given $\alpha : F \Rightarrow G$ a natural transformation between left adjoints $F \dashv R$ and $G \dashv Q$, and a morphism $f : B \to B'$, the reindexing functor $\mathscr{Mly}_{\mathcal{K}}(\alpha, f) : \mathscr{Mly}_{\mathcal{K}}(G, B) \to \mathscr{Mly}_{\mathcal{K}}(F, B')$ preserves all colimits –and thus, in the blanket assumption of presentability of $\mathcal{K}$, is a left adjoint; however, it fails to preserve limits (even terminal objects).[14]

[14] It is probably interesting to devise under which conditions the canonical map $\mathscr{Mly}_{\mathcal{K}}(\alpha, f)\big(\prod_{n\geq 1} Q^n B\big) \to \prod_{n\geq 1} R^n B'$, is well behaved in some sens (for example, under mild conditions that there exist at least one 'point' in its domain, the map is a split epi).

If $\mathcal{K}$ is monoidal its tensor functor $\_ \otimes - : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ now curries to the 'left regular representation' $\lambda : \mathcal{K} \to \mathsf{Cat}(\mathcal{K}, \mathcal{K}) : A \mapsto A \otimes -$ of $\mathcal{K}$ on itself, and as a consequence, we can pullback the total Mealy fibration and the total Moore fibration to obtain the left leg of the diagram

$$\begin{array}{ccc} \mathscr{Mly}^{\otimes}_{\mathcal{K}} & \longrightarrow & \mathscr{Mly}_{\mathcal{K}} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{K}^{\mathrm{op}} \times \mathcal{K} & \xrightarrow[\lambda^{\mathrm{op}} \times \mathcal{K}]{} & \mathsf{Cat}(\mathcal{K}, \mathcal{K})^{\mathrm{op}} \times \mathcal{K} \end{array} \tag{13}$$

which gives rise to the *monoidal Mealy* (two-sided) *fibration*

$$\mathcal{K} \xleftarrow{p^{\otimes}} \mathscr{Mly}^{\otimes}_{\mathcal{K}} \xrightarrow{q^{\otimes}} \mathcal{K} \tag{14}$$

(Similar considerations define $\mathscr{Mre}^{\otimes}_{\mathcal{K}}$, but we refrain from doing so for some technical reasons that make $\mathscr{Mly}^{\otimes}_{\mathcal{K}}$ a better-behaved object, cf. [15].) In fact, the terminology is chosen to inspire the fact that we have restricted the total Mealy category to the case where F-actions are monoidal and hint at the following result.

Proposition 6. *The monoidal Mealy fibration is a* monoidal two-sided fibration*, in the sense of [134,118], and the monoidal product interfiber is given by componentwise tensor product,*

$$\left(A, B; \left\| \tfrac{E}{d,s} \right\| \right) \otimes \left(A', B', \left\| \tfrac{E'}{d',s'} \right\| \right) = \left(A \otimes A', B \otimes B'; \left\| \tfrac{E \otimes E'}{d \otimes d', s \otimes s'} \right\| \right) \tag{15}$$

Theorem 3 ([76], [112, Def. 1] rephrased). *If $\mathcal{K}$ is* Cartesian *monoidal, the profunctor $\mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathsf{Cat}$ obtained from* (14) *carries the structure of a (pseudo)promonad, and it gives rise to a bicategory $\mathsf{Mly}_{\mathcal{K}}$ whose hom-categories are precisely the $\mathsf{Mly}_{\mathcal{K}}(A, B)$.*

The terminology introduced so far gives us enough leeway to introduce our first main theorem:

Theorem 4. *Let $(\mathcal{K}, \otimes, \partial)$ be a differential 2-rig; then the total categories of the monoidal Mealy and Moore fibrations are themselves differential 2-rigs for a universal choice of derivative functor $\bar{\partial} : \mathscr{Mly}^{\otimes}_{\mathcal{K}} \to \mathscr{Mly}^{\otimes}_{\mathcal{K}}$ such that the projection functors $p^{\otimes}, q^{\otimes}$ in* (14) *are (strict) morphisms of differential 2-rigs.*

Corollary 2. *The total category $\mathscr{Mly}^{\otimes}_{\mathsf{Spc}}$ obtained coupling Definition 7,* (13)*, and Proposition 6 is a differential 2-rig such that $\bar{\partial}$ commutes with all colimits that are preserved by ∂.*

Proposition 7. *The category $\mathscr{Mly}^{\otimes}_{\mathsf{Spc}}$ is locally presentable, so by the special adjoint functor theorem $\bar{\partial}$ has a left adjoint; in fact, more is true:*

- *the* fibration *of* (14) *is accessible (and cocomplete, hence locally presentable) in the sense of [90, 5.3.1], i.e. the total category $\mathscr{Mly}^{\otimes}_{\mathsf{Spc}}$ is locally presentable, the projection $\langle p, q \rangle$, all reindexing functors are accessible, and the pseudofunctor associated to the fibration preserves filtered colimits.*

– *the* $\bar{\partial}$ *functor is also continuous,* $(\mathcal{M}\ell y^{\otimes}_{\mathsf{Spc}}, \otimes_{\mathrm{Day}}, \bar{\partial})$ *is a* scopic *differential 2-rig.*

The following lemma splits the verification that $\mathsf{Mly}_{\mathcal{K}}$, defined in Definition 7, preserves filtered colimits in both components into two parts. The first part is a straightforward consequence of the fact that $\mathsf{Alg}(\_)$ preserves filtered colimits, in the sense that if $\mathcal{J}$ is a λ-filtered category, $\mathsf{Alg}(\mathrm{colim}_{\mathcal{J}} F_i) \cong \lim_{\mathcal{J}} \mathsf{Alg}(F_i)$. The key result allowing us to prove the second part is the fact that R as described in Equation 6 also preserves filtered colimits, hence for every filtered diagram, one has $T/\mathrm{colim}_{\mathcal{J}} B_i \cong \mathrm{colim}_{\mathcal{J}} \big(T/B_i\big)$.

Lemma 2. *For every fixed output object* $B \in \mathcal{K}$*, the functor* $\mathsf{Mly}_{\mathcal{K}}(-, B)$ *preserves filtered colimits. For every fixed dynamics* $F : \mathcal{K} \to \mathcal{K}$*, the functor* $\mathsf{Mly}_{\mathcal{K}}(F, -)$ *preserves filtered colimits.*

4 Differential and co/monadic dynamics

Besides usual monoidal automata, which have a distinguished differential flavour by the above remarks, one can exploit the other adjunction $\partial \dashv R$ where ∂ sits and look at categories $\mathsf{Mly}_{\mathsf{Spc}}(\partial, B)$ of *differential automata*, where dynamics are induced by the subsequent derivatives of a state object $E, \partial E, \dots, \partial^n E = E^{(n)}, \dots$.

Then, from every triple of adjoints $L \dashv \partial \dashv R$, 'monad-comonad' and 'comonad-monad' adjunctions $L\partial \dashv R\partial$ and $\partial L \dashv \partial R$ arise. One can then put the categories $\mathsf{Mly}_{\mathsf{Spc}}(L\partial, B)$ and $\mathsf{Mly}_{\mathsf{Spc}}(\partial L, B)$ under the spotlight using the language of Section 3.

This is, respectively, what we do in Section 5 below after we address the problem in more generality.

We want to study categories $\mathsf{Mly}_{\mathcal{K}}(T, B)$ of $(T \dashv S)$-automata where T is a left adjoint monad, and dually, categories $\mathsf{Mly}_{\mathcal{K}}(Q, B)$ of $(Q \dashv R)$-automata where Q is a left adjoint comonad.

In the case of a left adjoint monad, several technical results can be used to make the description of the categories $\mathsf{Mly}_{\mathcal{K}}(T, B)$ easier:

– [18, 4.3.2] if T is a left adjoint monad, with S as right adjoint comonad, its Eilenberg–Moore category $\mathcal{K}^T$ is cocomplete, with colimits preserved by the forgetful functor; in fact more is true:
– [18, 4.4.6] if T is a left adjoint monad, with S as right adjoint comonad, colimits in $\mathcal{K}^T$ are *created* by U, which in fact is comonadic and $\mathcal{K}^T$ identifies with the category of coEilenberg–Moore S-coalgebras.

The first general observation is completely elementary but already useful: considering that co/monads admit co/unit natural transformations to/from the identity functor, and given the functoriality of $\mathsf{Mly}_{\mathcal{K}}(-, B)$, we get canonical choices of functors

$$\mathsf{Mly}_{\mathcal{K}}(\mathrm{id}_{\mathcal{K}}, B) \longrightarrow \mathsf{Mly}_{\mathcal{K}}(Q, B) \qquad \mathsf{Mly}_{\mathcal{K}}(T, B) \longrightarrow \mathsf{Mly}_{\mathcal{K}}(\mathrm{id}_{\mathcal{K}}, B) \tag{16}$$

One can immediately prove by inspection that

Remark 15. The category $\mathsf{Mly}_{\mathcal{K}}(\mathrm{id}_{\mathcal{K}}, B)$ is the category of coalgebras for the functor $\_ \times B$.

Arguing again by functoriality, the monad structure on the functor T specifying the dynamics yields an augmented simplicial object [41], [129, 8.6]:

$$\mathsf{Mly}_{\mathcal{K}}(\boldsymbol{T},B)_\bullet = \left(\mathsf{Mly}_{\mathcal{K}}(\mathrm{id}_{\mathcal{K}},B) \xleftarrow{\eta^*} \mathsf{Mly}_{\mathcal{K}}(T,B) \underset{(T\eta)^*}{\overset{(\eta T)^*}{\leftleftarrows\!\!\xrightarrow{\mu^*}}} \mathsf{Mly}_{\mathcal{K}}(T^2,B) \cdots \right) \tag{17}$$

obtained feeding the bar resolution of T to the functor $\mathsf{Mly}_{\mathcal{K}}(-,B)$.

Dually, the cobar resolution of a left adjoint comonad Q yields an augmented cosimplicial object

$$\mathsf{Mly}_{\mathcal{K}}(\boldsymbol{Q},B)_\bullet = \left(\cdots \mathsf{Mly}_{\mathcal{K}}(Q^2,B) \underset{(\epsilon Q)^*}{\overset{(Q\epsilon)^*}{\leftleftarrows\!\!\xrightarrow{\sigma^*}}} \mathsf{Mly}_{\mathcal{K}}(Q,B) \xleftarrow[\epsilon^*]{} \mathsf{Mly}_{\mathcal{K}}(\mathrm{id}_{\mathcal{K}},B) \right) \tag{18}$$

Definition 8 (Bar and cobar Mealy complexes). *Both constructions are natural in the output object B, hence the above construction sets up functors $\mathsf{Mly}_{\mathcal{K}}(\boldsymbol{Q})_\bullet : \mathcal{K} \times \Delta \to \mathsf{Cat}$ and $\mathsf{Mly}_{\mathcal{K}}(\boldsymbol{T})_\bullet : \mathcal{K} \times \Delta^{\mathrm{op}} \to \mathsf{Cat}$. We refer to these as the* bar complex *of T-automata and the* cobar complex *of Q-automata.*

Remark 16. Let $\mathcal{K}$ be locally presentable. Given that $\mu^* : \mathsf{Mly}_{\mathcal{K}}(T,B) \to \mathsf{Mly}_{\mathcal{K}}(T^2,B)$ acts by precomposition with μ, sending $\left\|\frac{E}{d,s}\right\|$ to $\left\|\frac{E}{d\mu_E, s\mu_E}\right\|$ a swift application of the adjoint functor theorem yields a right adjoint μ_*.

Remark 17 (On monadic automata). It is reasonable to describe *Eilenberg–Moore*[15] Mealy automata, refining the pullbacks in Definition 5 by using the forgetful from $\mathcal{K}^T$ (the Eilenberg–Moore category of T) instead of $\mathsf{Alg}(T)$, and obtaining categories $\mu\mathsf{Mly}_{\mathcal{K}}(T,B)$ and $\mu\mathsf{Mre}_{\mathcal{K}}(T,B)$; in this case, some of the observations listed here carry over:

- $\mu\mathsf{Mly}_{\mathcal{K}}(\mathrm{id}_{\mathcal{K}},B)$ is just the slice $\mathcal{K}/B$, so the free-forgetful adjunction $F^T : \mathcal{K} \rightleftarrows \mathcal{K}^T : U^T$ induces a 'pulled-back' adjunction $\mu\mathsf{Mly}_{\mathcal{K}}(T,B) \rightleftarrows T/B$.
- Let S,T be monads on $\mathcal{K}$. Whenever a morphism of monads $\lambda : T \Rightarrow S$ in the sense of [8, §6.1] is given, the induced (colimit-preserving) functor $\mathcal{K}^S \to \mathcal{K}^T$ (cf. [*ibi*, Thm. 6.3]) induces in turn a (colimit-preserving) functor $\mu\mathsf{Mly}_{\mathcal{K}}(S,B) \to \mu\mathsf{Mly}_{\mathcal{K}}(T,B)$.

Remark 18. Working in the more restrictive case of Eilenberg–Moore automata is, however, rather unrewarding for a variety of reasons: first of all, there is the trivial remark that as soon as a carrier E has a structure $a : TE \to E$ of T-algebra, its 'dynamics' is pretty trivial, as a must be a split epi with a privileged right inverse η_E; thus, the composition $s \circ \eta_E$ 'knows everything' about the evolution of $\left\|\frac{E}{d,s}\right\|$. Second, the conditions for a natural transformation to induce functors between Eilenberg–Moore

[15] The 'Moore' of 'Moore automaton' and the Moore of 'Eilenberg–Moore' are two different people; the notion of 'Eilenberg–Moore Moore automaton' makes perfect sense as a category $\mathsf{Mre}_{\mathcal{K}}(T,B)$ arising as a pullback $\mathcal{K}^T \times_{\mathcal{K}} \mathcal{K}/B$. However, we leave Eilenberg–Moore Moore automata out of this note.

categories are fairly more imposing, and third, the morphisms inducing an analogue of (17),(18) are simply not available.

Something can be said, however, if we work 'interfiber' using Definition 6. A monad morphism in the sense of [117] induces a monad $\hat{S}$ on $\mathcal{K}^T$ so that the forgetful $U^T : \mathcal{K}^T \to \mathcal{K}$ is an intertwiner, hence leveraging on Definition 6 we can induce a functor

$$\mathsf{Mly}_{\mathcal{K}^T}(\hat{S},(B,b)) \longrightarrow \mathsf{Mly}_{\mathcal{K}}(S,B). \tag{19}$$

Dually, one can try to render the free functor $F_T : \mathcal{K} \to \mathcal{K}_T$ into the Kleisli category of T strong monoidal for a monoidal structure on $\mathcal{K}_T$; this will yield functors $\mathsf{Mly}_{\mathcal{K}}(S,B) \to \mathsf{Mly}_{\mathcal{K}_T}(\check{S}, F_T B)$. The matter is investigated in the second part of [51] when $F = A \otimes \_$. For example, consider $\mathcal{K}$ monoidal and with countable sums preserved by the tensor; then, every oplax monoidal monad $T : \mathcal{K} \to \mathcal{K}$ lifts a monoidal structure on $\mathcal{K}_T$ and one can then consider $\mathcal{K}_T$-valued F-machines, cf. [51, Prop. 30].

Remark 19 (On the proper choice of output objects). The construction of Definition 5 depends not only on F, but also on an output object B, usually thought as a 'space of responses' the machine $\left\|\frac{E}{d,s}\right\|$ can give as output. The choice of what B best models a given problem has to be made each time according to the nature of the problem itself. However, one is almost always led to consider choices of B that are 'spaces of truth values', like a Heyting or Boole algebra, or spaces of probabilities, like the closed unit interval $[0,1]$. The co/completeness of $\mathsf{Mly}_{\mathcal{K}}(F,B)$ and $\mathsf{Mre}_{\mathcal{K}}(F,B)$ established in Remark 9 entails that all algebraic structures (=all essentially algebraic theories) can be interpreted in such categories, and the nature of Spc as a presheaf topos entails that the construction of an object of internal real numbers is more or less straightforward. In particular,

- Hadamard Heyting/Boole algebra objects are just species $B : \mathsf{P} \to \mathsf{Set}$ which factor through the subcategory Heyt or Bool, the simplest case being the constant species $\mathfrak{B}$ at the booleans $\mathbf{B} = \{0 < 1\}$, with trivial action of each S_n ($\mathfrak{B}$ is the subobject classifier of Spc; another example of a Boolean algebra object in Spc is the species $\wp$ of subsets of Example 1.ES2);
- regarding Spc as a presheaf topos, it is easy to determine that the NNO, the object of integers, and of rationals, and of internal Dedekind reals [89, §VI.1] can be constructed as constant functors $c_{\mathbb{N}}, c_{\mathbb{Z}}, c_{\mathbb{Q}}, c_{\mathbb{R}}$ at natural, integers, rationals and reals in Set.

5 $L\partial$- and ∂L-algebras, the fourfold way

Remark 20 (On the structure of $L\partial$ and ∂L). Rajan [110] provides explicit formulas for the monad and comonad associated to $L \dashv \partial \dashv R$. Let $\mathfrak{F} : \mathsf{P} \to \mathsf{Set}$ be a species. Then,

- $L\partial\mathfrak{F}$ acts as $y[1] \otimes_{\mathrm{Day}} \partial\mathfrak{F}$; a structure of type $L\partial\mathfrak{F}$ on a finite set A chooses a point of A, and an $\mathfrak{F}$-structure on the complement of that point.
- $R\partial\mathfrak{F}$ acts as $A \mapsto \prod_{a\in A} \mathfrak{F}[(A \smallsetminus \{a\}) \sqcup \{\bullet\}]$, i.e. as $A \mapsto \mathfrak{F}A^A$; a structure of type $R\partial\mathfrak{F}$ on a finite set A chooses an $\mathfrak{F}$-structure on A for every $a \in A$. With a similar reasoning,

– $\partial L\mathfrak{F} = \partial(y[1] \otimes_{\text{Day}} \mathfrak{F})$ is the functor $\mathfrak{F} + L\partial\mathfrak{F}$;[16] Note in particular that the unit of the monad ∂L is the first coproduct injection.
– $\partial R\mathfrak{F}$ acts as $A \mapsto \mathfrak{F}[A]^A \times \mathfrak{F}[A] = R\partial\mathfrak{F}[A] \times \mathfrak{F}[A]$.

Both the following claims follow at once from the definition (and leave the question of when a generic scopic 2-rig admits a Euler derivation open).[17]

Remark 21 (The Euler derivation on Spc*).* The functor $L\partial = y[1] \otimes_{\text{Day}} \partial$ is a derivation in the sense of [86], and furthermore a left adjoint (with right adjoint $R\partial$), hence a colimit-preserving derivative functor.

Armed with these explicit descriptions, we can attempt to unveil the structure of the categories $\mathsf{Alg}(L\partial)$, $\mathsf{Alg}(\partial L)$, as building blocks for the category $\mathsf{Mly}_{\mathsf{Spc}}(L\partial, B)$, $\mathsf{Mly}_{\mathsf{Spc}}(\partial L, B)$. A thorough analysis of co/algebra structures for such interesting endofunctors of Spc seems to be missing from the existing literature. Rajan [110] goes as close as determining in painstaking detail the monad and comonad structures on $\partial L, \partial R, L\partial, R\partial$, but doesn't seem to provide a characterization for their endofunctor or Eilenberg–Moore algebras, or even for the (much easier, and somewhat more inspiring) bare endofunctor algebras. As one would expect from the adjunction relations $L\partial \dashv R\partial$ and $\partial L \dashv \partial R$ the structures of $L\partial$-algebras (=$R\partial$-coalgebras) and ∂L-algebras (=∂R-coalgebras) are tightly related. The following computations all follow a general argument, given Remark 20 a ∂L-algebra structure on a species F consists of a pair $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] : F + L\partial F \to F$ of maps $u : F \to F$ and $v : \partial F \to \partial F$ of endomorphisms, one for F and one for ∂F.

Example 6. A ∂L-algebra structure on the exponential species E reduces to a pair $u : E \to E$ and $v : LE \to E$, which in turn reduces to another endomap of E, given how E is a Napier object. Then, ∂L-algebra structures on E are representations of the free monoid $\mathbb{N}\langle d, c\rangle$ (cf. [53,54,55]) on 2 generators d, c over the set $E[1]$ (because endomaps of E are in bijection with elements of $E[1]$, by Yoneda). For set species, this must be trivial, for linear species this amounts to a 'character' for the monoid representation $\mathbb{N}\langle d, c\rangle$.

Example 7. For the species $\mathfrak{L}$ of linear orders, a ∂L-algebra map is a map $L \otimes L \to L$, since

$$\mathfrak{L} + L\partial(\mathfrak{L}) = \mathfrak{L} + y[1] \otimes_{\text{Day}} \mathfrak{L} \otimes_{\text{Day}} \mathfrak{L} \tag{20}$$

but then $\mathfrak{L} + y[1] \otimes_{\text{Day}} \mathfrak{L} \otimes_{\text{Day}} \mathfrak{L} = \mathfrak{L} \otimes_{\text{Day}} (1 + y[1] \otimes_{\text{Day}} \mathfrak{L})$, and the fact that $1 + y[1] \otimes_{\text{Day}} \mathfrak{L} \cong \mathfrak{L}$ is exactly the universal property satisfied by $\mathfrak{L}$ as initial algebra of $1 + y[1] \otimes_{\text{Day}} \_$.

[16] This gives rise to the evocative formula: $[\partial, L] = \partial L - L\partial = 1$, i.e. to the canonical commutation relation between position and momentum (up to a sign); in the language of *virtual species* [69,70,132,133] and [11, §2.5] such an equation can be made completely formal. As for its meaning, *hanc marginis exiguitas non caperet*, but see Problem 1 below.

[17] The differential operator $\Upsilon = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$ in $\mathbb{R}^n$ is called 'Euler homogeneity operator', cf. [35, p. 296]; another name for the same operation, 'numbering derivation', comes from Physics where if X^n represents something like a state of n bosons, like photons in a laser, then the differential operator $X \cdot D$ takes X^n to nX^n, where the coefficient 'counts' or 'numbers' the of bosons.

Example 8. A similar line of reasoning leads to the characterization of ∂L-algebra structures on the species of cycles, Example 1.ES5: since $\partial\mathfrak{Cyc} \cong \mathfrak{L}$, structures of ∂L-algebras are pairs, $\mathfrak{Cyc} \to \mathfrak{Cyc}$ and $\mathfrak{L} \to \mathfrak{L}$ of endomorphisms.

Example 9. For the species $\mathfrak{S}$ of permutations of Example 1, a ∂L-algebra structure consists of a pair $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] : \mathfrak{S} + L\partial\mathfrak{S} \to \mathfrak{S}$, where v can in turn be simplified into $\mathfrak{S} \otimes_{\text{Day}} (1 + y[1] \otimes_{\text{Day}} \mathfrak{L}) \cong \mathfrak{S} \otimes_{\text{Day}} \mathfrak{L}$ using Example 5.

6 Conclusions and future work

Problem 1. Let $\mathbf{K}$ be a strict 2-category with all finite weighted limits. Consider objects $X, B \in \mathbf{K}$ in a diagram of the following form:

$$X \xrightarrow[\text{id}_{\mathscr{X}}]{} X \xleftarrow[f]{} X \xrightarrow[f]{} X \xleftarrow[b]{} B \tag{21}$$

The *Vaucanson limit* [59][18] obtained from (21) consists of the limit obtained (cf. [28,78])

- the *inserter* $X \xleftarrow{u} \mathscr{I}(f, \text{id}_X) \xrightarrow{u} X$ of the left cospan;
- the *comma object* $X \xleftarrow{v} f/b \xrightarrow{q} B$ of the right cospan;
- the strict pullback $\mathscr{I}(f, \text{id}_X) \times_X (f/b)$ of u, v.

If $\mathbf{K}$ is the 2-category of categories, functors, and natural transformations, Vaucanson limits recover the categories $\mathsf{Mly}_{\mathscr{K}}(A, B)$ when $B = 1$ is the terminal category and b is an object therein.

Formal theory of Mealy automata is then the study of Vaucanson objects in $\mathbf{K}$. One can define analogues for $\mathsf{Mly}_{\mathscr{K}}(A, B)$, $\mathsf{Mre}_{\mathscr{K}}(A, B)$ enriched over a generic monoidal base $\mathscr{W}$ in the sense of [18, Ch. 6], [79], for example a quantale [113,26] like $[0, \infty]^{\text{op}}$, so that there is a *metric space* $\mathsf{Mly}_{(X,d)}(f, b)$ [81,21,61] associated to every nonexpansive map $f : X \to X$ and point $b \in X$. This begs the question: what is this theory, and how can it profit from being studied via discrete dynamical methods? Can it be related with fixpoint theory as classically intended in [46]?

Problem 2. The canonical commutation $[\partial, L] = \partial L - L\partial = 1$ valid in Joyal's virtual species suggests how L acts as a 'conjugate operator' to ∂. Compare this with the analogue relation $[x \cdot \_, \frac{d}{dx}] = 1$ valid in the ring $C^\omega(\mathbb{R})$ of analytic functions on, say, the real line [27, Ch. 5], [29]. Is it the case that there is a still undiscovered 'categorified Greenfunctionology' introducing a 'Heaviside distribution' Θ with the property that the colimit of F weighted by Θ is a solution of the differential equation $\partial G = F$ on species, i.e. $\partial \left(\int^X \Theta(X, \_) \times F[X] \right) \cong F$? Compare this with the well-known integral equation $\frac{d}{dx} \left(\int \Theta(x - t) f(t) dt \right) = f(x)$ for the Heaviside function, and cf. [23] where Day sketched a categorified theory of Fourier transforms (upper and lower transforms, Parseval relations, etc.) for categories enriched over a $*$-autonomous base $\mathscr{V}$ [9], generalizing Joyal's categories of analytic functors. We intend to pursue the matter, captivated by its compelling aesthetic beauty.

[18] Jacques de Vaucanson ($*$1709–†1782) was, besides the inventor of the modern lathe and of automatic loom, the creator of sophisticated and almost lifelike mechanical toys such as the '*flûteur automate*' and the '*canard défécateur*'. The mechanical duck appeared to have the ability to eat kernels of grain, and to metabolize and defecate them.

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