On the Centre of Strong Graded Monads

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Introduction.
The notions of *centre/centrality* and similarly *commutant/centraliser* can be formulated for many different kinds of algebraic structures, e.g. monoids, groups, semirings. It also makes sense in certain kinds of categorical settings. For instance, premonoidal categories admit a centre [4], which is essential for the development of the theory. Another example is given by enriched algebraic theories and associated monads which were shown to admit centralisers in [3]. In other related work [2], the authors study commutativity in a duoidal setting which is related to the above mentioned notions of centre and centralisers. More recently, in [1], the authors established some additional results for the centre of a strong monad on a symmetric monoidal category and also studied the more general concept of central submonad. Inspired by these developments, in this paper we consider yet another notion of centre, this time for strong monads that are graded by a (partially ordered) monoid, which can be seen as an immediate generalization of the definition of centre proposed in [1].

Constructing the centre of a monad is a way to recover the commutativity of it by removing the effects that violate it. In practice, commutativity of a monad is an important property which means that the effect can occur deterministically inside operations without forcing the evaluation order (most compilers are allowed to evaluate operands in the order of their choice for optimization purposes). A Graded monad, however, can be seen as a form of statistical analysis that gives insight on the effects that can occur. We think that studying the interaction between these structures is important.

First, we provide some computational intuition for the centre of a strong monad that may be useful to some readers. Strong monads on a symmetric monoidal category can be used to represent sequencing of computational effects. Informally, suppose we are given an effectful computation $f\colon X_1 \to X_2$ acting on some variable $x_1\colon X_1$ and another effectful computation $g\colon Y_1 \to Y_2$ acting on a different variable $y_1\colon Y_1$. Since these two computations are effectful, the two ways of sequencing the computations

$$\text{do } x_2 \leftarrow f(x_1); y_2 \leftarrow g(y_1); h(x_2, y_2)$$

and

$$\text{do } y_2 \leftarrow g(y_1); x_2 \leftarrow f(x_1); h(x_2, y_2)$$

do *not* necessarily have the same computational result, even though the two computations are acting on different variables that have different types. When we interpret the above computational situation in the Kleisli category $\mathbf{C}_T$ of a strong monad $T\colon \mathbf{C} \to \mathbf{C}$, this potential difference is reflected by the fact that $\mathbf{C}_T$ has a *premonoidal* structure [4], rather than a monoidal one, and the premonoidal product is not bifunctorial (in general), unlike the monoidal one. When the effect under consideration is commutative and so the monad T is also *commutative* (not just strong), then the two ways of sequencing above have the same computational result and this is reflected in the Kleisli category $\mathbf{C}_T$ which has a monoidal structure (not just premonoidal) in this case. So, we may naturally arrive at the notion of *centre* of a strong monad by identifying all the central elements/effects, i.e. those that commute with all other elements/effects. This is the approach taken in [1] and in this paper we consider a more general setting. Furthermore, we consider a wider range of effects, namely those that can be described by pomonoid-graded strong monads, and then we propose a definition for the centre of such monads:

Theorem 1 (Centre). *Let $\mathcal{G}$ be a pomonoid, if the $\mathcal{G}$-graded monad T on $\mathbf{C}$ is centralisable (for any object X in $\mathbf{C}$, for any element z in $\mathcal{Z}(\mathcal{G})$, a terminal graded central cone of T at (z, X) exists), then the assignment $\mathcal{Z}(-)$ extends to a commutative $Z(\mathcal{G})$-graded monad $(\mathcal{Z}, \eta^{\mathcal{Z}}, \mu^{\mathcal{Z}}, \tau^{\mathcal{Z}})$ on $\mathbf{C}$, called the* centre *of T. Moreover, $\mathcal{Z}$ is a commutative $\mathcal{Z}(\mathcal{G})$-graded submonad of T and the family of morphisms $\overset{z}{\iota}_X : \overset{z}{\mathcal{Z}}X \to \overset{z}{T}X$, determine a monomorphism of strong graded monads $\mathcal{Z} \to T$.*

The construction is analogous, but more general, compared to the one in [1]. More specifically, the premonoidal centre of graded monad is not yet defined, hence it is necessary to give a different proof by directly constructing the (graded) monad structure.

Then, we investigate possible research perspectives which exploit our understanding of the preliminary study performed here. In particular, we acknowledge the fact that the centre of graded monads is seldom usable in practice as it is even more constrained than that of a monad. Finally, we open the discussion to relaxations in which we may eventually obtain more refined notions of centres.

References

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