Optics, functorially

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Bidirectional transformations are mechanisms in software development that enable consistent interaction with data [5]. A well-known example is the concept of *lenses*, which consists of two methods: `get`, used to retrieve values from data, and `put`, employed to update existing data with a new value [7]. Recently, a wide variety of such bidirectional data accessors, including lenses, have been successfully formalised as *(mixed) optics* [4], using the categorical machinery of profunctors and of Tambara modules.
For simplicity, we shall work in the standard non-enriched setting. Let $\mathscr{M}$ be a monoidal category acting on two (small) categories $\mathscr{A}$ and $\mathscr{B}$ [1]. The category of optics $\mathbf{Optic}_{\mathscr{A},\mathscr{B}}$ has as objects pairs of objects of $\mathscr{A}$ and $\mathscr{B}$, respectively, and hom-sets

$$\mathbf{Optic}_{\mathscr{A},\mathscr{B}}((A',B'),(A,B)) = \int^{M\in\mathscr{M}} \mathscr{A}(A', M\bullet A)\times\mathscr{B}(M\bullet B, B')$$

The examples include cases where $\mathscr{A},\mathscr{B}$ are not necessarily small, but the above coend still exists. The monoidal actions of $\mathscr{M}$ on $\mathscr{A}$, respectively $\mathscr{B}$ represent the two different ways in which the categories $\mathscr{A},\mathscr{B}$, interact with the monoidal category $\mathscr{M}$: one when the data structure is decomposed, witnessed in the above coend by the hom-set $\mathscr{A}(A', M\bullet A)$, and another one, possibly different, when it is reconstructed, via $\mathscr{B}(M\bullet B, B')$. Lenses arise when $\mathscr{M}$ is a category with finite products and $\mathscr{A}=\mathscr{B}=\mathscr{M}$ with action given by the cartesian product.
It has been observed that the datum $(\mathscr{M},\mathscr{A},\mathscr{B})$ (actually, the actions of $\mathscr{M}$ on $\mathscr{A}$ and on $\mathscr{B}$ rather than the categories $\mathscr{A}$ and $\mathscr{B}$ themselves) induces a monad profunctor on $\mathscr{A}^{\mathsf{op}}\otimes\mathscr{B}$ [4, 8, 9], such that $\mathbf{Optic}_{\mathscr{A},\mathscr{B}}$ is the Kleisli object for this monad in the bicategory of profunctors $\mathbf{Prof}$. In particular, there is an identity-on-objects functor $\mathscr{A}^{\mathsf{op}}\otimes\mathscr{B}\longrightarrow\mathbf{Optic}_{\mathscr{A},\mathscr{B}}$. The $\mathscr{M}$-actions on $\mathscr{A}$ and on $\mathscr{B}$ extend to an action of $\mathscr{M}$ on optics, provided that $\mathscr{M}$ is symmetric monoidal [3].
In the case $\mathscr{A}=\mathscr{B}=\mathscr{M}$ and the actions were given by the underlying tensor product, it was shown that correspondence $\mathscr{M}\mapsto\mathbf{Optic}_{\mathscr{M},\mathscr{M}}$ becomes an endofunctor on the 2-category of symmetric monoidal categories and strong monoidal functors [9]. There is however a drawback. Identification of actions with the underlying monoidal product forbids any flexibility on optics, and does not comply with the plethora of existing examples [4]. How to preserve parametricity of optics in the two $\mathscr{M}$-actions, and in the same time, gain functoriality? Our main result provides an answer:

Theorem 1. *The correspondence* $(\mathscr{A},\mathscr{B})\mapsto\mathbf{Optic}_{\mathscr{A},\mathscr{B}}$ *determines a functor*

$$\mathbf{Optic}:\mathscr{M}\text{-}\mathbf{Act}_c\otimes\mathscr{M}\text{-}\mathbf{Act}_l\longrightarrow\mathbf{Cat}$$

from the Gray tensor product of the 2*-categories of* $\mathscr{M}$*-actions with lax, respectively colax* $\mathscr{M}$*-morphisms.*

In the domain of the **Optic** functor, on the level of 1-cells, instead of a single strong monoidal functor, we consider pairs of $\mathscr{M}$-morphisms of mixed variance, with the intuition that these *can act individually on each* $\mathscr{M}$*-action* of the optic rather than simultaneously on both. Such a pair (f,g) consists of functors $f:\mathscr{A}\longrightarrow\mathscr{A}'$ and $g:\mathscr{B}\longrightarrow\mathscr{B}'$, where f is an $\mathscr{M}$-colax morphism (equipped with a convenient natural transformation $\mathsf{cst}_{M,A}: f(M\bullet A)\longrightarrow M\bullet fA$), respectively g is an $\mathscr{M}$-lax morphism (with $\mathsf{st}_{M,B}: M\bullet gB\longrightarrow g(M\bullet B)$). The resulting functor between categories of optics $\mathbf{Optic}(f,g):\mathbf{Optic}_{\mathscr{A},\mathscr{B}}\longrightarrow\mathbf{Optic}_{\mathscr{A}',\mathscr{B}'}$ is induced by the composite

$$\mathscr{A}(A',M\bullet A)\otimes\mathscr{B}(M\bullet B,B')\to\mathscr{A}'(fA',f(M\bullet A))\otimes\mathscr{B}'(g(M\bullet B),gB')\to\mathscr{A}'(fA',M\bullet fA)\otimes\mathscr{B}'(M\bullet gB,gB')$$

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and commutes with the identity-on-objects functors $\mathscr{A}^{\mathsf{op}} \otimes \mathscr{B} \longrightarrow \mathbf{Optic}_{\mathscr{A},\mathscr{B}}$. If $\mathscr{M}$ is symmetric monoidal, $\mathbf{Optic}(f, g)$ also preserves the $\mathscr{M}$-action on optics.

Remark 2. *While lax $\mathscr{M}$-morphisms between $\mathscr{M}$-actions are a mild generalisation of strong functors on monoidal categories, the colax $\mathscr{M}$-morphisms have received much less attention in the literature (see however [2]), although they are not necessarily as rare as it might seem. We have observed that there is a one-to-one correspondence between co-pointed endofunctors on a cartesian category $\mathscr{M}$ and costrong ones (that is, colax with respect to the induced action of $\mathscr{M}$ on itself). In particular, comonads on cartesian categories are automatically costrong.*

Example 3. We would like to see now the **Optic** construction of Theorem 1 at work in the familiar case of lenses. Let $\mathscr{M}$ be a category with finite products and $\mathscr{A} = \mathscr{B} = \mathscr{M}$, with action given by the cartesian product. Then pairs (f, g) of $\mathscr{M}$-colax and lax morphisms make now sense if f is a costrong *endofunctor* on $\mathscr{M}$, and g is a strong one. Consider a lens $(\mathsf{get} : A' \longrightarrow A, \mathsf{put} : A' \times B \longrightarrow B')$. It transforms into a lens $(\mathsf{get}' : fA' \longrightarrow fA, \mathsf{put}' : fA' \times gB \longrightarrow gB')$, with $\mathsf{get}' = f\mathsf{get}$. However, put' does not look so simple. To gain some insight, recall from [6] that optics (in the general case) promote to a 2-category whose 2-cells explicitly keep track of the internal parameter M, and that in the cartesian case, there is a local adjunction between this 2-category of optics and the (discrete) 2-category of lenses, exhibiting the latter as a locally coreflective 2-subcategory of the former. Then $\mathbf{Optic}(f, g)$ is precisely a *morphism of adjunctions* between these.

A recent generalisation of (mixed) optics are dependent optics [10], where the two actions of the monoidal category $\mathscr{M}$ are replaced by a pair of pseudofunctors $\mathbf{B}^{\mathsf{op}} \longrightarrow \mathbf{Cat}$, where $\mathbf{B}$ is a bicategory. Taking $\mathbf{B}$ to be the delooping of $\mathscr{M}$ recovers usual optics. Theorem 1 extends to dependent optics where lax and colax $\mathscr{M}$-morphisms are replaced by lax and colax natural transformations.

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