Automata in W-Toposes, and General Myhill-Nerode Theorems*

Victor Iwaniack

Laboratoire J. A. Dieudonné, Université Côte d'Azur
`victor.iwaniack@unice.fr`

Abstract. We extend the functorial approach to automata by Colcombet and Petrişan [6] from the category of sets to any W-topos and establish general Myhill-Nerode theorems in our setting, including an explicit relationship between the syntactic monoid and the transition monoid of the minimal automaton. As a special case we recover the result of Bojańczyk, Klin and Lasota [4] for orbit-finite nominal automata by considering automata in the Myhill-Schanuel topos of nominal sets.

Keywords: Categories of machines · Topoi · Enriched Categories

Introduction

Automata theory appeared in the second half of the XXth century. Automata are simple formal machines meant to recognise languages, that is, sets of finite words over a finite alphabet. The fundamental theorem, first proven by Kleene [10] in 1956, characterises those languages that are recognised by *finite* automata. Two years later, Nerode [14] published another characterisation of this class of languages using an equivalence relation on words, closely related to a notion of a "minimal" automaton. More recently, Colcombet and Petrişan [6] gave an entirely categorical definition of automata, in particular making transparent the construction of the minimal automaton by means of purely categorical tools such as Kan extensions and (orthogonal) factorisation systems.

Our purpose here is to extend the categorical approach to automata theory by Colcombet and Petrişan [6] to more general contexts than those considered by the authors, namely to automata in an arbitrary W-topos. Topos theory is a far-reaching categorical generalisation of set theory with a strong topological flavour; the historic examples of toposes are those categories of sheaves over a topological space, but also categories of continuous actions of a topological group on discrete spaces. One of the notions crucial to the definition of an automaton in such a general context is "finiteness" and we will consider two different notions of finiteness which are well-established in topos theory: dK-finiteness (decidable

---

* This project has been partially funded by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No.670624).

Kuratowski finiteness) and decomposition-finiteness, both reducing to the classical notion of a finite set in the topos of sets. For both notions of finiteness we get a corresponding Myhill-Nerode Theorem characterising languages with a Nerode congruence of "finite type" as those recognised by "finite type" automata. The key property beneath these general Myhill-Nerode Theorems is the stability of "finite" objects under taking subquotients.

In Sections 1 and 2 we give definitions and proprieties about toposes and finiteness conditions we will consider. In Section 3 we enrich the functorial approach of Colcombet and Petrişan and use it to deduce Myhill-Nerode type theorems, and in the last sections we explore automata theory in specific toposes: toposes of G-sets for a discrete group G (Section 4) and automata in the Myhill-Schanuel topos of nominal sets (Section 5).

The main contribution here is the generalisation of the Colcombet and Petrişan framework, and the Myhill-Nerode theorems in other toposes than Set. Each theorem depends on a notion of "finiteness", and while Kuratowski finiteness is one of the most important notion of finiteness (for it is definable in any elementary topos and covers finite sets, finite coverings of topological spaces, and finite discrete group actions), it is not in general stable under subquotients, which is the key ingredient for those theorems.

Notations

We use the *diagrammatical order* for composition: if $f : A \to B$ and $g : B \to C$ are morphisms of some category, $fg : A \to C$ is their composition. By "factorisation system" we will always mean "orthogonal factorisation system" unless stated otherwise. If $\mathscr{D} \xleftarrow{L} \mathscr{C} \xleftarrow{R} \mathscr{D}$ is a diagram of functors, then we write $L \dashv R$ the fact that L is left adjoint to R and will usually denote the unit by $\eta : \mathrm{id}_{\mathscr{C}} \Rightarrow LR$ and the counit by $\varepsilon : RL \Rightarrow \mathrm{id}_{\mathscr{D}}$. For any morphism $a : Lc \to d$ of $\mathscr{D}$ we denote by $a^{\dashv} : c \to Rd$ of $\mathscr{C}$ its adjunct with respect to this adjunction, and for any morphism $b : c \to Rd$ of $\mathscr{C}$, $b^{\vdash} : Lc \to d$ of $\mathscr{D}$. Finally, if it exists, we denote by $\emptyset$ an initial object and $\mathbb{1}$ a terminal object.

1 Toposes

Definition 1. *An* (elementary) topos *is a category $\mathcal{E}$ with finite limits, exponentials (i.e. for each object B, the product by B endofunctor $(-) \times B$ has a right adjoint denoted by $(-)^B$ with counit $\mathrm{ev}^B : B \times (-)^B \Rightarrow \mathrm{id}_{\mathcal{E}}$) and a* subobject classifier*, that is a pointed object $\top : \mathbb{1} \to \Omega$ such that for each object A and subobject $S \hookrightarrow A$, there exists a unique morphism $\chi_S : A \to \Omega$ called the* characteristic map *such that the following diagram*

$$\begin{tikzcd} S \arrow[r, "!"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{1} \arrow[d, "\top"] \\ A \arrow[r, "\chi_S"'] & \Omega \end{tikzcd} \quad \textit{is a pullback.}$$

Example 1. Examples of toposes are categories $\mathbb{B}\, G := [G^{\mathrm{op}}, \mathrm{Set}]$ where G is a discrete group seen as a single-object groupoid, and $\mathbb{B}\, G$ is (equivalent to) the category of G-sets and equivariant functions: the subobject classifier is any two element set with trivial G action, exponentials Y^X are sets of mere functions $f : X \to Y$ endowed with the action $(f \cdot g)(x) = f(x \cdot g^{-1}) \cdot g$. For topological G, the category of continuous G-sets is also a topos, also denoted by $\mathbb{B}\, G$, cf. Mac Lane and Moerdijk [12, Section III.9]. Most famously, categories of sheaves $\mathrm{Sh}\, B$ over a topological space B (equivalent to the category of *étalé spaces over* B i.e. local homeomorphisms $p : E \to B$) are also toposes.

Remark 1. A topos is enriched over itself: the hom-objects are the exponentials, and we shall denote by $\mathrm{comp}_{A,B,C} : B^A \times C^B \to C^A$ the composition morphisms defined as adjuncts of $A \times B^A \times C^B \xrightarrow{\mathrm{ev}^A_B \times C^B} B \times C^B \xrightarrow{\mathrm{ev}^B_C} C$, and the identities by $\mathrm{id}^{\mathcal{E}}_A : \mathbb{1} \to A^A$ defined as the adjunct of $\mathrm{id}_A : A \to A$. Recall that exponentials define an internal hom functor $\mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{E}, (A, B) \mapsto B^A$ and for all $f : A \leftarrow A'$, $g : X \leftarrow X'$, $h : B \to B'$ and $a : A \times X \to B$ in $\mathcal{E}$:

$$((f \times g)ah)^{\dashv} = g(a^{\dashv})h^f .$$

We recall useful definitions on objects we will use:

Definition 2. *Let $\mathcal{E}$ be a topos and A any of its objects.*

1. *A subobject $S \leq A$ is* complemented *if there exists a subobject $C \leq A$ such that $S \cup C = A$ and $S \cap C = \emptyset$.*
2. *A is* decidable *if the diagonal $\Delta_A : A \xhookrightarrow{(\mathrm{id}_A, \mathrm{id}_A)} A \times A$ is complemented.*
3. *A is* connected *if it admits exactly two complemented subobjects $\emptyset$ and A.*

Amongst the examples cited, a space étalé over a base space B is connected in the topos $\mathrm{Sh}\, B$ iff it is connected as a topological space, and a G-set is connected iff transitive. Any G-set is decidable because $\mathbb{B}\, G$ is Boolean:

Definition 3. *A topos is*

1. Boolean *if every object is decidable (cf. Acuña-Ortega and Linton [1, Observation 2.6]);*
2. locally connected *if each object is a sum of connected objects;*
3. atomic *if Boolean and locally connected.*

A W-topos *is a topos admitting a free monoid $(\Sigma^*, m_\Sigma, \varepsilon_\Sigma)$ for every object Σ.*

Remark 2. The terminology “W-topos” comes from Moerdijk and Palmgren [13]. A topos is a W-topos iff it has a so-called *natural number object* (i.e. the free monoid generated by $\mathbb{1}$, cf. ibidem). A topos with countable coproducts is a W-topos because then $\Sigma^* := \sum_{n \in \mathbb{N}} \Sigma^n$.

Toposes $\mathbb{B}\, G$ are always atomic. A topos $\mathrm{Sh}\, B$ is locally connected iff B is as a topological space. Both types of toposes are W-toposes, because cocomplete.

2 Notions of Finiteness

Definition 4. *Let A be an object of an elementary topos $\mathcal{E}$. The submonoid of $(\Omega^A, \vee, \emptyset)$ generated by the singleton subobject $\{\cdot\}_A : A \hookrightarrow \Omega^A$ (adjunct of the characteristic morphism of the diagonal $\Delta_A : A \xhookrightarrow{(\mathrm{id}_A,\mathrm{id}_A)} A \times A$) is denoted by $K(A)$ and called the object of* Kuratowski-finite subobjects of A.

In toposes $\mathbb{B}\,G$ for a discrete group G, $K(A)$ is the set of finite subsets of A, endowed with the subset action $S \cdot g := \{s \cdot g | s \in S \subset A\}$ (cf. Example 1).

Definition 5. *Let $\mathcal{E}$ be a topos.*

1. *An object is called* decomposition-finite *if it is a finite coproduct of connected subobjects.*
2. *An object A is* Kuratowski-finite *or* K-finite *if the global element $\mathbb{1} \to \Omega^A$ corresponding to $A \leq A$ factors through $K(A) \leq \Omega^A$ (cf. Johnstone [7, Subsection D5.4]). It is* decidable Kuratowski finite *(*dK-finite*) if it is also decidable.*
3. *Let P be a non-empty class of* points *of $\mathcal{E}$ (left-adjoint left exact functors $x^* : \mathcal{E} \to \mathrm{Set}$), an object A is P-*stalkwise finite *if for all points x^* in P, $x^*(A)$ is a finite set.*

The dK-finite sheaves of $\mathrm{Sh}\,B$ are exactly finite coverings. Finiteness conditions in toposes $\mathbb{B}\,G$ will be discussed in Sections 4 and 5.

Proposition 1. *1. In an atomic topos, decomposition-finiteness is stable under taking subquotients.*
2. In a Boolean topos, dK-finiteness is stable under taking subquotients.
3. In any topos, stalkwise finiteness is stable under taking subquotients.

Proof. 1. If C and D are connected subobjects of a same object X, then either $C \cup D = \emptyset$, or $C \cup D \neq \emptyset$ is a complemented (because $\mathcal{E}$ is Boolean) non-empty subobject of connected C, therefore $C \cup D = C$ and by the same argument, $C \cup D = D$ so that $C = D$. This shows that if $\sum_{i \in I} A_i \leq \sum_{j \in J} B_j$ with connected A_i and B_j then $I \hookrightarrow J$ so that I is finite when J is. In any topos, homomorphic images of connected objects are connected, and this fact shows that decomposition-finiteness is stable under taking quotients.
2. According to Johnstone [7, Lemma 5.4.4.ii], in any topos, K-finite objects are stable under taking quotients, and if $\mathcal{E}$ is Boolean, they are moreover closed under taking subobjects (cf. Remark 5.4.20.ii⇒iii, ibidem).
3. Immediate because any point x^* preserves epimorphisms (as a left adjoint) and monomorphisms (as a left exact functor).

3 Automata in Toposes and Myhill-Nerode Theorems

Before enriching the approach of Colcombet and Petrişan [6], we recall their point of view. Consider a (complete deterministic) automaton (Q, i, F, δ) on an alphabet Σ (any set), meaning Q is a set of *states*, $i \in Q$ the *initial state*, $F \subset Q$ the set of *final states* and $\delta : Q \times \Sigma \to Q$ the *transition* function. The transition function gives, by iteration, a right action of the free monoid Σ^* generated by Σ on the set Q. In particular, we can interpret the action in a functorial (classical) way as a functor $\Sigma^* \to \mathrm{Set}$ where the monoid Σ^* is seen as a category with a single object st. Now the initial state $i \in Q$ can be seen as a global element $\mathbb{1} \to Q$, and the subset F of final states can be represented by its characteristic morphism $\chi_F : Q \to \Omega$ where Ω is the subobject classifier of Set, namely any two-element set of "truth values". All this data can be expressed by a functor $\mathcal{I}_\Sigma \to \mathrm{Set}$ with source freely generated by the quiver

$$\begin{tikzcd}
\mathrm{in} \arrow[r, "\triangleright"] & \mathrm{st} \arrow[loop above, "s\in\Sigma"] \arrow[r, "\triangleleft"] & \mathrm{out}
\end{tikzcd}$$

and the functor corresponding to the automaton (Q, i, F, δ)

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "i"] & Q \arrow[loop above, "{\delta(-,s), s\in\Sigma}"] \arrow[r, "\chi_F"] & \Omega
\end{tikzcd}$$

sends $(\mathrm{in}, \mathrm{st}, \mathrm{out})$ to $(\mathbb{1}, Q, \Omega)$, $\triangleright$ to the global element corresponding to i, $\triangleleft$ to the characteristic function corresponding to F, and extends the previous functor $\Sigma^* \to \mathrm{Set}$. This correspondence is in fact a bijection. The automata then organise as a (non-full) subcategory of a functor category. This approach merges the algebraic and coalgebraic point of view: an automaton seen as an algebra lacks terminal states, while an automaton seen as a coalgebra lacks an initial state.

Definition 6. *Let $\mathcal{E}$ be a W-topos, and fix Σ any object of this topos, we call an* alphabet*.*

- *A* language *on Σ is any subobject of Σ^*.*
- *A* (deterministic complete) automaton *on the alphabet Σ is a quadruple $\mathcal{A} = (Q, i, F, \delta)$ where*
  - *Q is the* states *object,*
  - *$i : \mathbb{1} \to Q$ is a global element, the* initial state*, of Q,*
  - *F is the subobject of Q of* final states*, which we identify with its characteristic morphism $\chi_F : Q \longrightarrow \Omega$, and*
  - *$\delta : Q \times \Sigma \to Q$ is the* transition morphism*.*

There is a notion of a language recognised by an automaton. To define it, observe that the adjunct $\delta^\dashv : \Sigma \to Q^Q$ of $\delta : Q \times \Sigma \to Q$, with respect to $(-) \times Q \dashv (-)^Q$, takes values in an internal monoid $(Q^Q, \mathrm{comp}_{Q,Q,Q}, \mathrm{id}^{\mathcal{E}}_Q)$. Because Σ^* is the free internal monoid, $\delta^\dashv$ extends uniquely to Σ^* into an internal monoid morphism we call $\delta^* : \Sigma^* \to Q^Q$.

Definition 7. *The language recognised by the automaton $\mathcal{A} = (Q, i, F, \delta)$ defined over Σ is the subobject $L(\mathcal{A})$ of Σ^* with characteristic morphism*

$$\Sigma^* \xrightarrow{\delta^*} Q^Q \xrightarrow{\chi_F^i} \Omega^{\mathbb{1}}.$$

*Automata $\mathcal{A}$ recognising a language L are called L-*automata*.*

3.1 Languages and Automata as Enriched Functors

The definitions we will give in the following subsections are immediate generalisations of the definitions of Colcombet and Petrişan [6], so that we will keep essentially the same terminology.

Definition 8. *Let Σ be an alphabet of a W-topos $\mathcal{E}$. The $\mathcal{E}$-category $\mathcal{I}_\Sigma$, called the* $\mathcal{E}$-category of internal behaviours over the alphabet Σ*, is the $\mathcal{E}$-category freely generated by the $\mathcal{E}$-quiver Q_Σ with three vertices* in, st *and* out*, and objects of edges $Q_\Sigma(\mathrm{in}, \mathrm{st}) = \mathbb{1} = Q_\Sigma(\mathrm{st}, \mathrm{out})$, $Q_\Sigma(\mathrm{st}, \mathrm{st}) = \Sigma$ and $Q_\Sigma(X, Y) = \emptyset$ for every other cases. Spelt out, $\mathcal{I}_\Sigma$ is defined by:*

Objects in*,* st *and* out*.*
Objects of morphisms *given by the table:*

$\mathcal{I}_\Sigma(\downarrow, \rightarrow)$	in	st	out
in	$\mathbb{1}$	Σ^*	Σ^*
st	$\emptyset$	Σ^*	Σ^*
out	$\emptyset$	$\emptyset$	$\mathbb{1}$

Composition morphisms *being of the following form, depending on the domain and codomain of definition:*

- $\Sigma^* \times \Sigma^* \xrightarrow{m_\Sigma} \Sigma^*$*,*
- $\mathbb{1} \times \Sigma^* \cong \Sigma^*$*,*
- $\Sigma^* \times \mathbb{1} \cong \Sigma^*$ *and*
- *if the source is $\emptyset$ or the target is $\mathbb{1}$, then the composition is trivial.*

Proposition 2. *Automata $\mathcal{A} = (Q, i, F, \delta)$ over Σ are in bijective correspondence with $\mathcal{E}$-functors $\underline{\mathcal{A}} : \mathcal{I}_\Sigma \to \mathcal{E}$ sending $(\mathrm{in}, \mathrm{out})$ to $(\mathbb{1}, \Omega)$.*

Proof. We use the fact that $\mathcal{I}_\Sigma$ is a free $\mathcal{E}$-category: $\mathcal{E}$-functors $\underline{\mathcal{A}}$ sending $(\mathrm{in}, \mathrm{out})$ to $(\mathbb{1}, \Omega)$ correspond to $\mathcal{E}$-quiver morphisms $\alpha : Q_\Sigma \to \mathcal{E}$ sending $(\mathrm{in}, \mathrm{out})$ to $(\mathbb{1}, \Omega)$, which correspond to automata (Q, i, F, δ) over Σ:

- α takes $(\mathrm{in}, \mathrm{st}, \mathrm{out})$ to $(\mathbb{1}, Q, \Omega)$
- $\alpha_{\mathrm{in},\mathrm{st}} = i : \mathbb{1} \to Q$
- $\alpha_{\mathrm{st},\mathrm{out}} = \ulcorner F \urcorner : \mathbb{1} \to \Omega^Q$ adjunct of $\chi_F : Q \cong \mathbb{1} \times Q \to \Omega$
- $\alpha_{\mathrm{st},\mathrm{st}} = \delta^{\dashv} : \Sigma \to Q^Q$ adjunct of $\delta : Q \times \Sigma \to Q$.

Definition 9. *The full sub-$\mathcal{E}$-category of $\mathcal{I}_\Sigma$ spanned by the objects* in *and* out *is denoted by $\mathcal{O}_\Sigma \overset{\iota_\Sigma}{\hookrightarrow} \mathcal{I}_\Sigma$ and called the* $\mathcal{E}$-category of observable behaviours over the alphabet Σ*.*

Remark 3. All the non-trivial data of an $\mathcal{E}$-functor $F : \mathcal{O}_\Sigma \to \mathrm{Set}$ sending $(\mathrm{in}, \mathrm{out})$ to $(\mathbb{1}, \Omega)$ is contained in $F_{\mathrm{in},\mathrm{out}} : \Sigma^* \to \Omega$, therefore such functors bijectively correspond to languages L over Σ and we denote by $\underline{L}$ the corresponding functor. This bijection provides a notion of language recognised by an automaton-as-a-functor which is coherent with the associated automaton as shown in the next proposition.

Proposition 3. *Under the bijection of Proposition 2, the restriction of an automaton* $\underline{\mathcal{A}} : \mathcal{I}_{\varSigma} \to \mathcal{E}$ *to the sub-*$\mathcal{E}$*-category* $\mathcal{O}_{\varSigma}$ *corresponds to the language* $L(\mathcal{A})$ *recognised by the automaton* $\mathcal{A}$ *corresponding to* $\underline{\mathcal{A}}$ *i.e.* $\underline{L(\mathcal{A})} = \underline{\mathcal{A}}\big|_{\mathcal{O}_{\varSigma}}$.

Proof. Let $\mathcal{A} = (Q, i, F, \delta)$ be an automaton over $\varSigma$, then

$$L(\mathcal{A}) \coloneqq \varSigma^* \xrightarrow{\delta^*} Q^Q \xrightarrow{\chi_F^i} \varOmega^{\mathbb{1}}$$

corresponds through the natural bijection $\mathcal{E}(\varSigma^*, \varOmega^{\mathbb{1}}) \cong \mathcal{E}(\mathbb{1} \times \varSigma^*, \varOmega)$ to

$$\begin{aligned}
&\mathbb{1} \times \varSigma^* \xrightarrow{i \times \varSigma^*} Q \times \varSigma^* \xrightarrow{\delta^{*\vdash}} Q \xrightarrow{\chi_F} \varOmega \text{ (cf. Remark 1)} \\
=&\mathbb{1} \times \varSigma^* \xrightarrow{\varepsilon_{\varSigma} \underline{\mathcal{A}}_{\text{in,st}} \times \varSigma^*} Q \times \varSigma^* \xrightarrow{\underline{\mathcal{A}}^{\vdash}_{\text{st,st}}} Q \xrightarrow{\chi_F} \varOmega \text{ by definition of } \underline{\mathcal{A}} \\
=&\mathbb{1} \times \varSigma^* \xrightarrow{\varepsilon_{\varSigma} \times \varSigma^*} \varSigma^* \times \varSigma^* \xrightarrow{\underline{\mathcal{A}}_{\text{in,st}} \times \varSigma^*} Q \times \varSigma^* \xrightarrow{\underline{\mathcal{A}}^{\vdash}_{\text{st,st}}} Q \xrightarrow{\chi_F} \varOmega \\
=&\mathbb{1} \times \varSigma^* \xrightarrow{\varepsilon_{\varSigma} \times \varSigma^*} \varSigma^* \times \varSigma^* \xrightarrow{m_{\varSigma}} \varSigma^* \xrightarrow{\underline{\mathcal{A}}_{\text{in,st}}} Q \xrightarrow{\chi_F} \varOmega \\
=&\mathbb{1} \times \varSigma^* \xrightarrow{\sim} \varSigma^* \xrightarrow{\underline{\mathcal{A}}_{\text{in,st}}} Q \xrightarrow{\chi_F} \varOmega
\end{aligned}$$

where $(\underline{\mathcal{A}}_{\text{in,st}} \times \varSigma^*)\underline{\mathcal{A}}^{\vdash}_{\text{st,st}} = m_{\varSigma}\underline{\mathcal{A}}_{\text{in,st}}$ by $\mathcal{E}$-functoriality of $\underline{\mathcal{A}}$, and $(\varepsilon_{\varSigma} \times \varSigma^*)m_{\varSigma} = \mathbb{1} \times \varSigma^* \xrightarrow{\sim} \varSigma^*$ by left unitality in the monoid $\varSigma^*$. Then the morphism above is

$$\mathbb{1} \times \varSigma^* \xrightarrow{\sim} \varSigma^* \xrightarrow{\underline{\mathcal{A}}_{\text{in,st}}} Q \xrightarrow{(\varepsilon_F \underline{\mathcal{A}}_{\text{st,out}})^{\vdash}} \varOmega$$

by definition of $\underline{\mathcal{A}}$ and corresponds through $\mathcal{E}(\mathbb{1} \times \varSigma^*, \varOmega) \cong \mathcal{E}(\mathbb{1}, \varOmega^{\varSigma^*})$ to

$$\begin{aligned}
&\mathbb{1} \xrightarrow{\varepsilon_{\varSigma}} \varSigma^* \xrightarrow{\underline{\mathcal{A}}_{\text{st,out}}} \varOmega^Q \xrightarrow{\varOmega^{\underline{\mathcal{A}}_{\text{in,st}}}} \varOmega^{\varSigma^*} \\
=&\mathbb{1} \xrightarrow{\varepsilon_{\varSigma}} \varSigma^* \xrightarrow{m_{\varSigma}{}^{\dashv}} \varSigma^{*\varSigma^*} \xrightarrow{\underline{\mathcal{A}}_{\text{in,out}}} \varOmega^{\varSigma^*} \text{ by } \mathcal{E}\text{-functoriality of } \underline{\mathcal{A}} \\
=&\mathbb{1} \xrightarrow{\text{id}^{\mathcal{E}}_{\varSigma^*}} \varSigma^{*\varSigma^*} \xrightarrow{\underline{\mathcal{A}}_{\text{in,out}}} \varOmega^{\varSigma^*} \text{ by right unitality in the monoid } \varSigma^*
\end{aligned}$$

and finally going through the inverse natural bijection $\mathcal{E}(\mathbb{1}, \varOmega^{\varSigma^*}) \cong \mathcal{E}(\mathbb{1} \times \varSigma^*, \varOmega) \cong \mathcal{E}(\varSigma^*, \varOmega^{\mathbb{1}})$, the morphism above is $\underline{\mathcal{A}}_{\text{in,out}} : \varSigma^* \to \varOmega^{\mathbb{1}}$.

3.2 Categories of Automata

We follow Colcombet and Petrişan [6] definition of automaton morphism, i.e. the morphisms we consider are coalgebra morphisms that respects the initial state.

Definition 10. *Let* $\underline{L} : \mathcal{O}_{\varSigma} \to \mathcal{E}$ *be a language over* $\varSigma$ *in a W-topos* $\mathcal{E}$*. The category* Auto(L) *of* L-automata *has as objects the* $\mathcal{E}$*-functors extending* $\underline{L}$ *along the inclusion* $\iota_{\varSigma} : \mathcal{O}_{\varSigma} \hookrightarrow \mathcal{I}_{\varSigma}$*, and as morphisms* $\mathcal{E}$*-natural transformations* $\alpha : \underline{\mathcal{A}} \Rightarrow \underline{\mathcal{B}}$ *restricting to* $\text{id}_{\underline{L}}$ *on* $\mathcal{O}_{\varSigma}$*.*

In some cases we will obtain an automaton recognising the language only up to an automorphism of $\varOmega$, so that $\underline{\mathcal{A}}\big|_{\mathcal{O}_{\varSigma}}$ might only be isomorphic to $\underline{L}$.

Lemma 1 (Strictification of an automaton with respect to a language). *Let* $\underline{L}$ *be a language and* $\underline{\mathcal{A}}$ *an automaton, both defined over an alphabet* Σ*. If there exists an* $\mathcal{E}$*-natural isomorphism* $\varphi : \underline{\mathcal{A}}\big|_{\mathcal{O}_\Sigma} \cong \underline{L}$*, then there exists an automaton* $\underline{\mathcal{B}} \in \mathrm{Auto}(L)$ *isomorphic as an* $\mathcal{E}$*-functor to* $\underline{\mathcal{A}}$ *via* $\psi : \underline{\mathcal{A}} \cong \underline{\mathcal{B}}$ *such that* $\iota_\Sigma * \psi = \varphi$*. The automaton* $\underline{\mathcal{B}}$ *is constructed as the* $\mathcal{E}$*-functor* $\underline{\mathcal{A}}$ *with this only difference:*

$$\underline{\mathcal{B}}_{\mathrm{st,out}} = \Sigma^* \xrightarrow{\mathcal{A}_{\mathrm{st,out}}} \Omega^Q \xrightarrow{(\varphi_{\mathrm{out}}^{-1})^Q} \Omega^Q.$$

It was one of the main insights of Colcombet and Petrişan [6, (2.2)] that the minimal automaton recognising a given language can be constructed by factorising the canonical map from the initial automaton to the final automaton. This remains true in our enriched context as we will see in the next subsections.

3.3 Initial and Terminal Automata as Enriched Kan Extensions

In our setting, automata are (strict) extensions of languages seen as $\mathcal{E}$-functors along the fully faithful $\mathcal{E}$-functor $\iota_\Sigma : \mathcal{O}_\Sigma \to \mathcal{I}_\Sigma$. But in the case of (pointwise) Kan extensions along a fully faithful functor, the Kan extensions are genuine extensions up to isomorphism, and therefore provide two automata:

Proposition 4. *In a W-topos* $\mathcal{E}$*, the initial* $\emptyset(L)$ *and terminal* $\mathbb{1}(L)$ *automata exist for any language* L *over any alphabet* Σ*; they are respectively the left and the right* $\mathcal{E}$*-enriched Kan extension of* $\underline{L} : \mathcal{O}_\Sigma \to \mathcal{E}$ *along the fully faithful* $\mathcal{E}$*-functor* $\iota_\Sigma : \mathcal{O}_\Sigma \hookrightarrow \mathcal{I}_\Sigma$*. They can be explicitly computed as follows:*

$$\begin{array}{rl|ll}
\Sigma^* = & \emptyset(L)(\mathrm{st}) & \mathbb{1}(L)(\mathrm{st}) & = \Omega^{\Sigma^*} \\
\mathrm{id}_{\Sigma^*} = & \emptyset(L)_{\mathrm{in,st}} & \mathbb{1}(L)_{\mathrm{in,st}} & = (m_\Sigma \chi_L)^{\dashv} : \Sigma^* \to \Omega^{\Sigma^*} \\
{\Sigma^*}^{\Sigma^*} \leftarrow \Sigma^* : {m_\Sigma}^{\dashv} = & \emptyset(L)_{\mathrm{st,st}} & \mathbb{1}(L)_{\mathrm{st,st}} & = ((\Omega^{m_{\Sigma^*}})^{\vdash})^{\vdash} : \Sigma^* \to (\Omega^{\Sigma^*})^{\Omega^{\Sigma^*}} \\
\Omega^{\Sigma^*} \leftarrow \Sigma^* : (m_\Sigma \chi_L)^{\dashv} = & \emptyset(L)_{\mathrm{st,out}} & \mathbb{1}(L)_{\mathrm{st,out}} & = (\chi_{\ni_{\Sigma^*}})^{\dashv} : \Sigma^* \to \Omega^{\Omega^{\Sigma^*}}
\end{array}$$

where $\mathbb{1}(L)_{\mathrm{st,st}}$ *is obtained from* Ω^{m_Σ} *by the natural bijection* $\mathcal{E}(\Omega^{\Sigma^*}, \Omega^{\Sigma^* \times \Sigma^*}) \cong \mathcal{E}(\Omega^{\Sigma^*}, (\Omega^{\Sigma^*})^{\Sigma^*}) \cong \mathcal{E}(\Omega^{\Sigma^*} \times \Sigma^*, \Omega^{\Sigma^*}) \cong \mathcal{E}(\Sigma^*, (\Omega^{\Sigma^*})^{\Sigma^*})$

As for Colcombet and Petrişan [6, Lemma 2.9], the Kan extensions, now enriched, are computed pointwise; but here the formula (cf. Kelly [9, (2.2) with (4.24)]) only uses exponentials (resp. binary products), finite products (resp. coproducts, finite because $\mathcal{I}_\Sigma$ has only three objects) and an equaliser (resp. coequaliser), which all exist in $\mathcal{E}$.

Proposition 5. *In a W-topos* $\mathcal{E}$*, let* $\underline{\mathcal{A}}$ *be an* L*-automaton over an alphabet* Σ*, the unique automaton morphism* α *from* $\emptyset(L)$ *to* $\underline{\mathcal{A}}$ *is given by*

$$\alpha_{\mathrm{st}} = \underline{\mathcal{A}}_{\mathrm{in,st}} : \Sigma^* \to Q$$

and the unique automaton morphism β *from* $\underline{\mathcal{A}}$ *to* $\mathbb{1}(L)$ *by*

$$\beta_{\mathrm{st}} = (Q \times \Sigma^* \xrightarrow{\underline{\mathcal{A}}^{\vdash}_{\mathrm{st,out}}} \Omega)^{\dashv} : Q \to \Omega^{\Sigma^*} \ \mathit{wrt.}\ \mathcal{E}(Q \times \Sigma^*, \Omega) \cong \mathcal{E}(Q, \Omega^{\Sigma^*}).$$

In particular, the unique morphism from $\emptyset(L)$ *to* $\mathbb{1}(L)$ *is*

$$(m_\Sigma \chi_L)^{\dashv} : \Sigma^* \to \Omega^{(\Sigma^*)}.$$

3.4 Minimal Automaton

We understand minimality with respect to a factorisation system, cf. Colcombet and Petrişan [6, Subsection 2.2]:

Definition 11. *In a category $\mathscr{C}$ endowed with a factorisation system (E, M), we say an object X (E, M)*-divides *an object Y if there exists a span*

$$X \xleftarrow{e \in E} Z \xrightarrow{m \in M} Y$$

in $\mathscr{C}$. An object is minimal *if it divides any object of $\mathscr{C}$.*

We recall the key idea of Colcombet and Petrişan [6, Lemma 2.3] to compute the minimal automaton:

Proposition 6. *Let $\mathscr{C}$ be a category with a factorisation system (E, M). If $\mathscr{C}$ has an initial and a terminal object, then the object through which the unique arrow from the initial to the terminal object (E, M)-factorises is (E, M)-minimal.*

However, here, we need the factorisation system to be on the category of automata which is a category of enriched functors. Therefore, we have to lift the $(\mathrm{epi}, \mathrm{mono})$ factorisation system on $\mathcal{E}$ to the $\mathcal{E}$-functors category $[\mathcal{I}_\Sigma, \mathcal{E}]$. Given two $\mathcal{V}$-functors and a $\mathcal{V}$-natural transformation $\alpha : \mathcal{F} \Rightarrow \mathcal{G}$, the pointwise factorisation of α according to a given factorisation system, might only give an unenriched functor. Thus, we need the factorisation to have more properties, which leads to the definition of an enriched factorisation system.

Definition 12. *Let $\mathcal{V}$ be a symmetric closed monoidal category, and $\mathscr{C}$ a $\mathcal{V}$-category. A factorisation system (E, M) on $\mathscr{C}$ is $\mathcal{V}$*-enriched *if for all $A \xleftarrow{e \in E} B$ and $X \xrightarrow{m \in M} Y$, the following square*

$$\begin{tikzcd}
\mathscr{C}(A, X) \arrow[r, "\mathscr{C}(A,m)"] \arrow[d, "\mathscr{C}(e,X)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{C}(A, Y) \arrow[d, "\mathscr{C}(e,Y)"] \\
\mathscr{C}(B, X) \arrow[r, "\mathscr{C}(B,m)"'] & \mathscr{C}(B, Y)
\end{tikzcd}$$

is a pullback in $\mathcal{V}$.

One can characterise enriched factorisation systems amongst unenriched ones using powers or copowers, according to Lucyshyn-Wright [11, Theorem 5.7]:

Proposition 7. *If $\mathscr{C}$ has $\mathcal{V}$-copowers (respectively $\mathcal{V}$-powers), then a factorisation system (E, M) on $\mathscr{C}$ is enriched if and only if E is stable under $\mathcal{V}$-copowers (resp. M is stable under $\mathcal{V}$-powers). This is in particular the case if $\mathcal{E} = \mathscr{C}$ is a topos, and (E, M) is the $(\mathrm{epi}, \mathrm{mono})$ factorisation system.*

An enriched factorisation system ensures that factorising enriched natural transformations provides an enriched functor. Proposition 7 can be used to show:

Proposition 8. *Let (E, M) be a $\mathcal{V}$-factorisation system on $\mathcal{V}$, and $\mathscr{I}$ a small $\mathcal{V}$-category. Then the classes $E_{\mathscr{I}}$ of $\mathcal{V}$-natural transformations that are pointwise in E, and $M_{\mathscr{I}}$ of $\mathcal{V}$-natural transformations that are pointwise in M form a $\mathcal{V}$-enriched factorisation system on $[\mathscr{I}, \mathcal{V}]$.*

A lifted factorisation system on $[\mathcal{I}_\Sigma, \mathcal{E}]$ can then be restricted to $\mathrm{Auto}(L)$:

Proposition 9. *Any $\mathcal{E}$-factorisation system on $\mathcal{E}$ can be lifted to* $\mathrm{Auto}(L)$*, so that the factorisation of an automaton morphism is obtained as the pointwise factorisation of the underlying $\mathcal{E}$-natural transformation.*

Proof. Consider a morphism $\alpha : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ and its factorisation $\underline{\mathcal{A}} \xrightarrow{e} F \xrightarrow{m} \underline{\mathcal{B}}$ in $[\mathcal{I}_\Sigma, \mathcal{E}]$. Then by unicity of the factorisation in $\mathcal{E}$ we have:

$$\begin{tikzcd}
 & F(\mathrm{in}) \arrow[dr, "m_{\mathrm{in}}"] \arrow[dd, "\sim"] & & & & F(\mathrm{out}) \arrow[dr, "m_{\mathrm{out}}"] \arrow[dd, "\sim"] & \\
\mathbb{1} \arrow[ur, "e_{\mathrm{in}}"] \arrow[dr, "\mathrm{id}_{\mathbb{1}}"'] & & \mathbb{1} & \text{and} & \Omega \arrow[ur, "e_{\mathrm{out}}"] \arrow[dr, "\mathrm{id}_{\Omega}"'] & & \Omega \\
 & \mathbb{1} \arrow[ur, "\mathrm{id}_{\mathbb{1}}"'] & & & & \Omega \arrow[ur, "\mathrm{id}_{\Omega}"'] &
\end{tikzcd}$$

because $\alpha_{\mathrm{in}} = \mathrm{id}_{\mathrm{in}} = e_{\mathrm{in}} m_{\mathrm{in}}$ and $\alpha_{\mathrm{out}} = \mathrm{id}_{\mathrm{out}} = e_{\mathrm{out}} m_{\mathrm{out}}$ so that $F\big|_{\mathcal{O}_\Sigma} \cong \underline{L}$ and by Lemma 1 we can find a factorisation of α through an L-automaton.

The category $\mathrm{Auto}(L)$ is now canonically endowed with the pointwise $(\mathrm{epi}, \mathrm{mono})$ factorisation system.

Definition 13. *Let L be a language over an alphabet Σ in a W-topos $\mathcal{E}$. The automaton* $\mathrm{Min}(L)$ *through which the unique arrow from the initial automaton to the terminal automaton factorises with respect to pointwise* (*epi, mono*) *factorisation system is the* minimal automaton *of L.*

The name "minimal automaton" designates this particular construction but it is really minimal in the sense of Definition 11:

Corollary 1. *Let L be a language on an alphabet Σ in a W-topos $\mathcal{E}$. The minimal automaton of L is a subquotient of any automaton that recognises L.*

Proof. By Proposition 9 and 4 the category $\mathrm{Auto}(L)$ has pointwise $(\mathrm{epi}, \mathrm{mono})$ factorisation system, initial and terminal objects, therefore Proposition 6 ensures $\mathrm{Min}(L)$ is minimal in this context i.e. $\mathrm{Min}(L)$ is a subquotient of every L-automata.

3.5 Internal Nerode Congruence

In Set, the *Nerode congruence of a language L over Σ* is an equivalence relation on words over Σ defined by $u \sim_L v$ iff for all words x, $ux \in L \Leftrightarrow vx \in L$. It is strictly the same as saying $u \sim_L v$ iff $u^{-1}L = v^{-1}L$. Then the Nerode congruence is merely the kernel pair of left division of L, $u \mapsto u^{-1}L$, which in turn is the adjunct of $(u, w) \mapsto \chi_L(uw)$, namely the composite of monoid multiplication of Σ^* with χ_L.

Definition 14. *Let Σ be an alphabet in a W-topos $\mathcal{E}$. The* Nerode congruence[1] of a language L over Σ *is the kernel pair $\equiv_L$ of*

$$(m_\Sigma \chi_L)^{\dashv} : \Sigma^* \to \Omega^{\Sigma^*}$$

where m_Σ is the multiplication of the free monoid Σ^*.

Proposition 10. *In a W-topos $\mathcal{E}$, the states object of* $\mathrm{Min}(L)$ *is the quotient of Σ^* by the internal Nerode congruence.*

Proof. By Corollary 1, $\emptyset(L) \xrightarrow{!} \mathbb{1}(L)$ factorises as

$$\emptyset(L) \hookrightarrow \mathrm{Min}(L) \twoheadrightarrow \mathbb{1}(L)$$

and therefore we have an image factorisation of $\emptyset(L)(\mathrm{st}) \xrightarrow{!_{\mathrm{st}}} \mathbb{1}(L)(\mathrm{st})$

$$\emptyset(L)(\mathrm{st}) \hookrightarrow \mathrm{Min}(L)(\mathrm{st}) \twoheadrightarrow \mathbb{1}(L)(\mathrm{st})$$

and the morphism above is $(m_\Sigma \chi_L)^{\dashv}$ according to Proposition 5. A topos $\mathcal{E}$ is regular and in a regular category, the kernel pair of a morphism is canonically isomorphic to its image, so that $\mathrm{Min}(L)(\mathrm{st}) \cong \Sigma^*_{\equiv_L}$.

3.6 Myhill-Nerode Theorems for Different Finiteness Conditions

The following Myhill-Nerode theorems have two main cases of application: the first is in Set, the classical Myhill-Nerode theorem stating that a language is regular if and only if the Nerode congruence is of finite index (cf. Nerode [14]), and the second in the topos Nom of *nominal sets*, proven for any G-sets topos by Bojańczyk, Klin and Lasota [4] (where G is a discrete group or G is the topological group of permutations of natural numbers acting on discrete spaces), which states that a G-language (resp. nominal language) is regular, in the sense that it is recognised by an orbit-finite deterministic G-automaton (resp. nominal automaton) if and only if the quotient of the nominal set of words on the alphabet by the Nerode congruence is orbit-finite. Our Theorem 1 is a generalisation and another point of view on Bojańczyk, Klin and Lasota [4, Theorems 3.8 and 5.2].

Definition 15. *Each time we consider a finiteness condition (FC), we say an automaton* is (FC) *if its states object is (FC). A language L is* (FC)-regular *if it admits an (FC) automaton that recognises it.*

Theorem 1. *Let L be a language on an alphabet Σ in a W-topos $\mathcal{E}$.*

1. *For any non-empty class of points P of $\mathcal{E}$, L is P-stalkwise-regular iff $\Sigma^*_{/\equiv_L}$ is P-stalkwise finite.*
2. *If $\mathcal{E}$ atomic, L is decomposition-regular iff $\Sigma^*_{/\equiv_L}$ is decomposition-finite.*
3. *If $\mathcal{E}$ is Boolean, L is K-regular iff $\Sigma^*_{/\equiv_L}$ is K-finite.*

---

[1] It is not an internal monoid congruence, it is only a categorical congruence, namely an internal equivalence relation.

Proof. By Corollary 1, $\mathrm{Min}(L)$ exists and by Proposition 10, $\mathrm{Min}(L)(\mathrm{st}) = \Sigma^*_{/\equiv_L}$. If there is some L-automaton $\underline{\mathcal{A}}$ that is (FC) for one of those cases, then because $\mathrm{Min}(L)$ divides $\underline{\mathcal{A}}$, $\mathrm{Min}(L)(\mathrm{st}) = \Sigma^*_{/\equiv_L}$ divides $\underline{\mathcal{A}}(\mathrm{st})$ which is (FC), so by Proposition 1, $\Sigma^*_{/\equiv_L}$ is also (FC).

3.7 The Syntactic Monoid

There exists an algebraic notion of recognition where the recogniser is a monoid morphism (cf Jean-Éric Pin [15, Subsection IV.2.1]). With this point of view, automata are merely presentations of such algebraic recognisers, given by the transition monoid of the automaton. Amongst monoids recognising a language there is a smallest recogniser with respect to monoid divisibility: the syntactic monoid of a language. It can be defined abstractly as the quotient of the monoid of words by a *syntactic* congruence, or simply by the fact it is the transition monoid of the minimal automaton. We will now describe this (non-functorial) construction in any W-topos and discuss its behaviour with respect to a given finiteness condition.

Definition 16. *Let L be a language on an alphabet Σ in a W-topos $\mathcal{E}$. We say a monoid morphism $\varphi : \Sigma^* \to M$* recognises *L if there exist $\chi : M \to \Omega$ making the following triangle commute:*

$$\begin{tikzcd} \Sigma^* \arrow[r, "\chi_L"] \arrow[d, "\varphi"'] & \Omega \\ M \arrow[ur, "\chi"'] & \end{tikzcd}$$

A monoid M recognises *L if there exists such a φ with target M.*
We call the triple (M, φ, χ) an L-monoid*, and an* L-monoid morphism *from (M, φ, χ) to (M', φ', χ') is a monoid morphism $f : M \to M'$ such that those two triangles commute:*

$$\begin{tikzcd} & \Sigma^* \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ M \arrow[rr, "f"'] & & M' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\chi"'] & & M' \arrow[dl, "\chi'"] \\ & \Omega & \end{tikzcd}$$

in other words: f is a morphism from φ to φ' in $\Sigma^*/\mathrm{Mon}(\mathcal{E})$ and from χ to χ' in $\mathcal{E}/\Omega$ as well.
This defines a category $\mathrm{Mon}(L)$, and we denote by $\Sigma\,\mathrm{Mon}(L)$ the full subcategory of Σ-generated L-monoids *spanned by L-monoids of the form $(M, \varphi, \chi_{\varphi(L)})$ where φ is epic in $\mathcal{E}$, and for those we usually drop the now implicit $\chi_{\varphi(L)}$.*

Remark 4. If χ classifies $p : P \hookrightarrow M$ and χ' classifies $q : Q \hookrightarrow M'$, then we have $f\chi' = \chi$ iff $p = f^*(q)$ where f^* is the inverse image of (i.e. pullback along) f.

Lemma 2. *If $\varphi : \Sigma^* \to M$ recognises L, then $(\mathrm{Im}\,\varphi, e : \Sigma^* \twoheadrightarrow \mathrm{Im}\,\varphi, \chi_{\varphi(L)})$ is a Σ-generated L-monoid, where e is the image of φ and $\chi_{\varphi(L)}$ is the characteristic morphism of the image inclusion of the composite $L \hookrightarrow \Sigma^* \xrightarrow{\varphi} \mathrm{Im}\,\varphi$ in $\mathrm{Im}\,\varphi$.*

Proof. By pasting law of pullbacks, because the outer rectangle is a pullback ($\varphi\chi = \chi_L$ so by Remark 4, $L = \varphi^*(P)$) and the right one two,

$$\begin{tikzcd}
L \arrow[rrd, bend left, "!"] \arrow[rd, "\exists!"] \arrow[dd, hook] & & \\
& P \arrow[r, "!"] \arrow[dd, hook] \arrow[rdd, phantom, very near start, "\lrcorner"] & \mathbb{1} \arrow[dd, "\top"] \\
\Sigma^* \arrow[rd, "\varphi"'] & & \\
& M \arrow[r, "\chi"'] & \Omega
\end{tikzcd}$$

then the left one is also a pullback and by (epi, mono) factorisation of φ and pulling back along the inclusion of P we have a unique filler (by functoriality of factorisation systems)

$$\begin{tikzcd}
L \arrow[r, two heads] \arrow[d, hook] & \varphi(L) \arrow[r, hook] \arrow[d, hook, "\exists!"] & P \arrow[d, hook] \\
\Sigma^* \arrow[r, two heads] & \mathrm{Im}\,\varphi \arrow[r, hook] & M
\end{tikzcd}$$

which is also a monomorphism as a pullback of a monomorphism. Then this diagram provides the (epi, mono) factorisation of $L \hookrightarrow \Sigma^* \xrightarrow{\varphi} M$, and because all the squares here are pullbacks we have

$$\begin{tikzcd}
L \arrow[r, two heads] \arrow[d, hook] \arrow[rd, dashed, "(t)", "\circlearrowleft"'] & \varphi(L) \arrow[r, "!"] \arrow[d, hook] & \mathbb{1} \arrow[d, "\top"] \\
\Sigma^* \arrow[r, two heads] & \mathrm{Im}\,\varphi \arrow[r, "\chi_{\varphi(L)}"'] & \Omega
\end{tikzcd}$$

showing the commutativity of the triangle (t) by universal property of Ω.

Consider an L-monoid morphism $f : (M, \varphi, \chi) \to (N, \psi, \chi')$ with φ and ψ epimorphic in $\mathcal{E}$. We have to show that $f\chi_{\psi(L)} = \chi_{\varphi(L)}$. For this consider the following diagram

$$\begin{tikzcd}
& L \arrow[r, equal] \arrow[d, two heads] \arrow[rd, phantom, "\circlearrowleft"] & L \arrow[d, two heads] & \\
L \arrow[ru, equal, "\circlearrowleft"'] \arrow[r, two heads] \arrow[d, hook] \arrow[rd, phantom, "(1)"] & \varphi(L) \arrow[r, two heads] \arrow[d, hook] \arrow[rd, phantom, "(2)"] & \psi(L) \arrow[r, "!"] \arrow[d, hook] \arrow[rd, phantom, "(3)"] & \mathbb{1} \arrow[d, "\top"] \\
\Sigma^* \arrow[r, two heads, "\varphi"'] & M \arrow[r, two heads, "f"'] & N \arrow[r, "\chi_{\psi(L)}"'] & \Omega
\end{tikzcd}$$

where the vertical composite morphisms $L \to M$ and $L \to N$ are the (epi, mono) factorisations of, respectively, $i\varphi$ and $i\psi$ (where $i : L \hookrightarrow \Sigma^*$ is the inclusion of L in Σ^*), and $\varphi(L) \to \psi(L)$ is the unique filler. Showing $f\chi_{\psi(L)} = \chi_{\varphi(L)}$ amount to showing the square (2+3) is a pullback (by Remark 4). But we already know that (3) is a pullback so we have to show, by pullback pasting, that (2) is a pullback, and to do so we use Carboni, Janelidze, Kelly and Paré [5, Lemma

4.6]. As a topos $\mathcal{E}$, is regular, φ is an epimorphism (by hypothesis) and (1) is a pullback ($\varphi\chi_{\varphi(L)} = \chi_L$ then L is the pullback of $\varphi(L)$ along φ, cf. Remark 4); thus (2) is a pullback iff (1+2) is. But $\varphi f = \psi$ therefore (cf Remark 4 again) (1+2) is indeed a pullback, translating $\psi\chi_{\psi(L)} = \chi_L$.

Of course, Σ^* always recognises any language L. This can be seen as the consequence of the fact that the initial automaton always exists.

Definition 17. *Let* $\underline{\mathcal{A}} : \mathcal{I}_\Sigma \to \mathrm{Set}$ *be an automaton on an alphabet* Σ *in a W-topos* $\mathcal{E}$*. The morphism* $\underline{\mathcal{A}}_{\mathrm{st},\mathrm{st}} : \Sigma^* \to \underline{\mathcal{A}}(\mathrm{st})^{\underline{\mathcal{A}}(\mathrm{st})}$ *is a monoid morphism with image factorisation* $\Sigma^* \overset{\tau_{\mathcal{A}}}{\twoheadrightarrow} T(\mathcal{A}) \hookrightarrow \underline{\mathcal{A}}(\mathrm{st})^{\underline{\mathcal{A}}(\mathrm{st})}$*. The monoid* $T(\mathcal{A})$ *is called the* transition *monoid of* $\mathcal{A}$*.*

Proposition 11. *If* $\underline{\mathcal{A}}$ *recognises* L*, then* $(T(\underline{\mathcal{A}}), \tau_{\underline{\mathcal{A}}})$ *is a* Σ*-generated* L*-monoid.*

Proof. We apply Lemma 2 to the following triangle

$$\begin{tikzcd}
 & \Sigma^* \arrow[dl, "\underline{\mathcal{A}}_{\mathrm{st},\mathrm{st}}"'] \arrow[dr, "\underline{\mathcal{A}}_{\mathrm{in},\mathrm{out}}=\chi_L"] & \\
\underline{\mathcal{A}}(\mathrm{st})^{\underline{\mathcal{A}}(\mathrm{st})} \arrow[rr, "((\varepsilon_\Sigma \underline{\mathcal{A}}_{\mathrm{st},\mathrm{out}})^{\vdash})^{\varepsilon_\Sigma \underline{\mathcal{A}}_{\mathrm{in},\mathrm{st}}}"'] & & \Omega
\end{tikzcd}$$

which commutes because of $\mathcal{E}$-functoriality of $\underline{\mathcal{A}}$ and the fact that ε_Σ is the identity of st in the $\mathcal{E}$-category $\mathcal{I}_\Sigma$.

Remark 5. However, the T construction is not functorial; to witness this in Set, consider any finite automaton $\mathcal{A}$ with at least two distinct states q and r on an alphabet with at least two letters a and b, and construct an automaton $\mathcal{B}$ by adding a new state t to $\mathcal{A}$, and such that $\underline{\mathcal{B}}(a)(t) = q$, $\underline{\mathcal{B}}(b)(t) = r$, $\underline{\mathcal{B}}(c)(t) = t$ if $c \in \Sigma \setminus \{a, b\}$ and $\underline{\mathcal{B}}(c)(s) = \underline{\mathcal{A}}(c)(s)$ if $c \in \Sigma, s \in \underline{\mathcal{A}}(\mathrm{st})$. The initial state and final states of $\mathcal{B}$ are those of $\mathcal{A}$ so that the inclusion of states of $\mathcal{A}$ in those of $\mathcal{B}$ defines a monomorphic automaton morphism from $\mathcal{A}$ to $\mathcal{B}$, and the transition monoid of $\mathcal{B}$ contains strictly more endofunctions than those of $\mathcal{A}$. However, an L-monoid morphism between $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ has to be surjective because $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ are, which is impossible in that case.

The transition monoid construction might not be functorial but it at least preserves divisibility.

Proposition 12. *The* T *construction induces two functors : a covariant one* $\overrightarrow{T}$ *from the wide subcategory* $\mathrm{Auto}_{epi}(L)$ *of* $\mathrm{Auto}(L)$ *of automata and pointwise epic automaton morphisms to* $\Sigma\,\mathrm{Mon}(L)$*, and a contravariant one* $\overleftarrow{T}$ *from the wide subcategory* $\mathrm{Auto}_{mono}(L)$ *of* $\mathrm{Auto}(L)$ *of automata and pointwise monic automaton morphisms to* $\Sigma\,\mathrm{Mon}(L)$*.*

Proof. Consider a pointwise epimorphic automaton morphism $e : \underline{\mathcal{A}} \twoheadrightarrow \underline{\mathcal{B}}$. By $\mathcal{E}$-naturality of e, epimorphy of e entailing monomorphy of $\underline{\mathcal{B}}(\mathrm{st})^e$ and $(\mathrm{epi}, \mathrm{mono})$ factorisation we have a unique filler

$$\begin{tikzcd}
\Sigma^* \arrow[r, two heads, "\tau_{\underline{\mathcal{A}}}"] \arrow[d, equal] & T(\underline{\mathcal{A}}) \arrow[rr, hook] \arrow[d, two heads, "\exists ! T(e)"] & & \underline{\mathcal{A}}(\mathrm{st})^{\underline{\mathcal{A}}(\mathrm{st})} \arrow[d, "e^{\underline{\mathcal{A}}(\mathrm{st})}"] \\
\Sigma^* \arrow[r, two heads, "\tau_{\underline{\mathcal{B}}}"'] & T(\underline{\mathcal{B}}) \arrow[r, hook] & \underline{\mathcal{B}}(\mathrm{st})^{\underline{\mathcal{B}}(\mathrm{st})} \arrow[r, hook, "\underline{\mathcal{B}}(\mathrm{st})^{e}"'] & \underline{\mathcal{B}}(\mathrm{st})^{\underline{\mathcal{A}}(\mathrm{st})}
\end{tikzcd}$$

making the diagram commute, and it also is an epimorphism. By functoriality of orthogonal factorisation systems, this construction is functorial where it makes sense, namely on $\mathrm{Auto}_{\mathrm{epi}}(L)$. A similar argument of functorial factorisation allows constructing the functor $\overleftarrow{T}$ and uses the fact that monomorphy of m implies monomorphy of $m^{\underline{\mathcal{A}}(\mathrm{st})}$.

Corollary 2. *If* $\underline{\mathcal{A}}$ *divides* $\underline{\mathcal{B}}$*, i.e. we have* $\underline{\mathcal{A}} \xleftarrow{e} \underline{\mathcal{C}} \xhookrightarrow{m} \underline{\mathcal{B}}$*, then* $T(\underline{\mathcal{A}}) \xtwoheadleftarrow{\overrightarrow{T}(e)} T(\underline{\mathcal{C}}) \xtwoheadleftarrow{\overleftarrow{T}(m)} T(\underline{\mathcal{B}})$*, so in particular,* $T(\underline{\mathcal{A}})$ *divides* $T(\underline{\mathcal{B}})$*.*

An automaton recognising L can be seen as a presentation of an L-monoid. But in fact, each Σ-generated L-monoid can be seen canonically as the transition monoid of an automaton.

Lemma 3. *The covariant functor* $\overrightarrow{T} : \mathrm{Auto}_{epi}(L) \to \Sigma\,\mathrm{Mon}(L)$ *has a section (up to natural isomorphism)* A *defined by*

$$\underline{A(M,\varphi)}(\mathrm{st}) = M,\ \underline{A(M,\varphi)}_{\mathrm{in,st}} = \varphi,\ \underline{A(M,\varphi)}_{\mathrm{st,st}} = \Sigma^* \xrightarrow{\varphi} M \xrightarrow{m^{\dashv}} M^M,$$

$$\underline{A(M,\varphi)}_{\mathrm{st,out}} = \Sigma^* \xrightarrow{\varphi} M \xrightarrow{m^{\dashv}} M^M \xrightarrow{\chi^M_{\varphi(L)}} \Omega^M$$

and sends an L*-monoid morphism* $f : (M,\varphi) \to (N,\psi)$ *to an automaton morphism* $(\mathrm{id}_{\mathbb{1}}, f, \mathrm{id}_{\Omega}) : \underline{A(M,\varphi)} \to \underline{A(N,\psi)}$*.*

Proof. Denote by (M,m,e) the monoid M, $\underline{A(M,\varphi)}_{\mathrm{in,out}} = (\underline{A(M,\varphi)}_{\mathrm{in,st}} \times \varepsilon_{\Sigma}\underline{A(M,\varphi)}_{\mathrm{st,out}})\,\mathrm{ev}^M_{\Omega}$ by $\mathcal{E}$-functoriality, but $\varepsilon_{\Sigma}\underline{A(M,\varphi)}_{\mathrm{st,out}} := \varepsilon_{\Sigma}\varphi m^{\dashv}\chi^M_L = em^{\dashv}\chi_L$ because φ is a monoid morphism and $em^{\dashv} = \mathrm{id}^{\mathcal{E}}_M$ by left unitality in the monoid M. Therefore, $\underline{A(M,\varphi)}_{\mathrm{in,out}} = (\varphi \times \chi^{\dashv}_L)\,\mathrm{ev}^M_{\Omega} : \Sigma^* \times \mathbb{1} \to \Omega$ corresponds to $\varphi\chi_{\varphi(L)} : \Sigma^* \to \Omega$, itself equal to χ_L because (M,φ) is a Σ-generated L-monoid. We have to show that M is isomorphic to a submonoid of M^M in a natural way; it is sort of an internal Cayley theorem. Recall that by definition $\Sigma^* \xtwoheadrightarrow{\tau_{\underline{A(M,\varphi)}}} T(\underline{A(M,\varphi)}) \hookrightarrow M^M$ is the (regular epi, mono)-factorisation of the monoid morphism $\underline{A(M,\varphi)}_{\mathrm{st,st}} = \varphi m^{\dashv}$ which is also a factorisation : φ is epic by hypothesis and $m^{\dashv}$ is monic because it has a retract M^e by left unitality.

Theorem 2. *Let* L *be a language over an alphabet* Σ *in a W-topos* $\mathcal{E}$*. The transition monoid of the minimal automaton* $T(\mathrm{Min}(L))$ *is minimal in the category of* L*-monoids. We then call this monoid the* syntactic monoid *of* L *and denote it by* $\mathrm{Syn}(L)$*.*

Proof. Let (M,φ,χ) be any L-monoid, and by Lemma 2, $(\mathrm{Im}\,\varphi, e)$ its reflected Σ-generated sub-L-monoid.

$$\begin{tikzcd}
\mathrm{Min}(L) & {} \arrow[d, mapsto, "\overrightarrow{T}"] & \underline{\mathcal{B}} \arrow[ll, two heads] \arrow[rr, hook] & {} \arrow[d, mapsto, "\overleftarrow{T}"] & \underline{A(\mathrm{Im}\,\varphi, e)} & & \text{in } \mathrm{Auto}(L) \\
\mathrm{Syn}(L) & {} & T(\underline{\mathcal{B}}) \arrow[ll, two heads] & {} & (\mathrm{Im}\,\varphi, e) \arrow[ll, two heads] \arrow[r, hook] & (M, \varphi, \chi) & \text{in } \mathrm{Mon}(L)
\end{tikzcd}$$

In Set, the syntactic monoid provides another characterisation of regularity: a language is regular if and only if its syntactic monoid is finite. We discuss this fact with different finiteness conditions. In the following results we might use the fact that $\mathrm{Min}(L)$ is a *reachable automaton.*

Definition 18. *An L-automaton $\underline{\mathcal{A}}$ is* reachable *if $\emptyset(L) \xrightarrow{!} \underline{\mathcal{A}}$ is an epimorphism.*

According to Proposition 5, $\underline{\mathcal{A}}$ is then reachable iff $\underline{\mathcal{A}}_{\mathrm{in,st}}$ is an epimorphism.

Lemma 4. *If $\underline{\mathcal{A}}$ is reachable, the states object of $\underline{\mathcal{A}}$ is (canonically) a quotient of its transition monoid.*

Proof. Consider the diagram

$$\begin{tikzcd}
\Sigma^* \arrow[r, two heads, "\underline{\mathcal{A}}_{\mathrm{in,st}}"] \arrow[d, two heads, "\tau_{\underline{\mathcal{A}}}"'] \arrow[dr, "\underline{\mathcal{A}}_{st,st}"'] & \underline{\mathcal{A}}(\mathrm{st}) \\
T(\underline{\mathcal{A}}) \arrow[r, hook] & \underline{\mathcal{A}}(\mathrm{st})^{\underline{\mathcal{A}}(\mathrm{st})} \arrow[u, "\underline{\mathcal{A}}(\mathrm{st})^{\varepsilon\,\Sigma\,\underline{\mathcal{A}}_{\mathrm{in,st}}}"']
\end{tikzcd}$$

where the top-right triangle commute by $\mathcal{E}$-functoriality of $\underline{\mathcal{A}}$ and the bottom-left one does by definition of $\tau_{\underline{\mathcal{A}}}$. Then, the whole diagram commutes and therefore by left-cancellation of epimorphisms, the composite arrow $T(\underline{\mathcal{A}}) \hookrightarrow \underline{\mathcal{A}}(\mathrm{st})^{\underline{\mathcal{A}}(\mathrm{st})} \to \underline{\mathcal{A}}(\mathrm{st})$ is an epimorphism.

Theorem 3. *Let L be a language in a W-topos $\mathcal{E}$.*

1. *If* $\mathrm{Syn}(L)$ *is K-finite then L is K-regular.*
2. *If $\mathcal{E}$ is Boolean and L is K-regular then* $\mathrm{Syn}(L)$ *is K-finite.*

Proof.
1. By Lemma 4, $\mathrm{Min}(L)(\mathrm{st})$ is a quotient of $\mathrm{Syn}(L)$; the former is K-finite when the latter is according to Johnstone [7, Lemma 5.4.4.ii].
2. As $\mathcal{E}$ is Boolean, $\mathrm{Min}(L)$ is K-finite because is K-regular by Theorem 1, and also by Booleanity, exponentials and subobjects of K-finite objects are K-finite according to Acuña-Ortega and Linton [1, Main Theorem], therefore $\mathrm{Min}(L)^{\mathrm{Min}(L)}$ is finite and so is its subobject $\mathrm{Syn}(L)$.

4 Equivariant automata

Automata in toposes $\mathbb{B}G$ for a discrete group G were explored by Bojańczyk, Klin and Lasota [4, Section 3] with decomposition-finiteness. Here we also discuss dK-finiteness in this setting.

Definition 19. *An equivariant automaton is an automaton in a topos* $\mathbb{B}\,G$ *for* G *a discrete group.*

Proposition 13. *Let* G *be a discrete group. An object* A *of* $\mathbb{B}\,G$ *is*

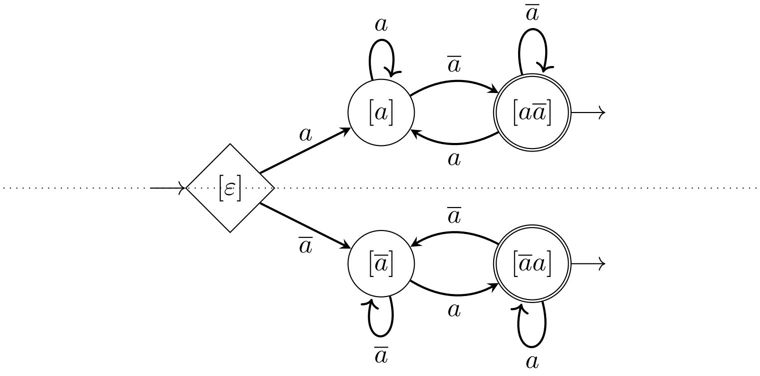
1. *dK-finite iff it is finite as a set, and*
2. *decomposition-finite iff it has a finite number of orbits.*

Proof. 1. See Johnstone [7, Example 5.4.19].
2. Connected G-sets are exactly the transitive ones, therefore orbits.

Example 2. As a first "toy" example we consider an automaton in the topos of sets with an involution, namely $\mathbb{Z}_{/2\,\mathbb{Z}}\,\mathrm{Set}$, the topos of the actions of the two-element group. For each set with an involution (X, i) we shall denote $i(x) = \overline{x}$. Consider the two-letter alphabet $\Sigma = \{a, \overline{a}\}$ where the involution exchanges the two letters. The free (internal) monoid is simply the free monoid Σ^* where the involution swaps the two letters. We define the language $L = \left\{l u \bar{l} \middle| l \in A, u \in A^*\right\}$ of words of length at least two whose first and last letters are different. The Nerode quotient $\Sigma^*_{/\equiv_L}$ is the five elements set

$$\left\{L, a^{-1}L, \overline{a}^{-1}L, (a\overline{a})^{-1}L, (\overline{a}a)^{-1}L\right\}$$

so that its only fixed point is $L \in \Sigma^*_{/\equiv_L}$. This allows us to describe the minimal $\mathbb{Z}\,/2\,\mathbb{Z}\,\mathrm{Set}$-automaton of this $\mathbb{Z}\,/2\,\mathbb{Z}\,\mathrm{Set}$-language:



5 Nominal Automata

We define the Myhill-Schanuel topos $\mathbb{B}\,\mathrm{Aut}(\mathbb{N})$ of *nominal sets* and *equivariant functions* to be the category of continuous actions on discrete spaces of the topological group $\mathrm{Aut}(\mathbb{N})$ of permutations of a countable set $\mathbb{N}$ of *names*, where the topology is induced by the inclusion $\mathrm{Aut}(\mathbb{N}) \subset \mathbb{N}^{\mathbb{N}}$ (where $\mathbb{N}^{\mathbb{N}}$ carries the product topology); equivariant functions are those functions that commute with the action. According to Mac Lane and Moerdijk [12, Theorem 3.9.2], this indeed form an atomic W-topos.

Definition 20. *A nominal automaton is an automaton in the topos* $\mathbb{B}\,\mathrm{Aut}(\mathbb{N})$.

In the topos $\mathbb{B}\,\mathrm{Aut}(\mathbb{N})$, point 2 of Theorem 1 becomes:

Theorem 4 (Cf. Bojańczyk, Klin and Lasota [4, Theorem 5.2]). *Let L be a language over Σ in the topos $\mathbb{B}\,\mathrm{Aut}(\mathbb{N})$. The language L is decomposition-regular iff $\Sigma^*_{/\equiv_L}$ is orbit-finite.*

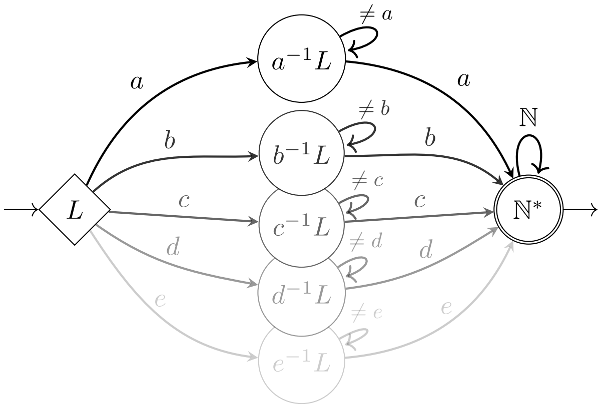
Example 3. Consider on the alphabet $\mathbb{N}$ the classical example of the language L of words where the first letter appears at least once again further in the word:

$$L = \{ab_1b_2\cdots b_n \in \mathbb{N}^* | n \in \mathbb{N}, a, b_i \in \mathbb{N}, \exists i \in \mathbb{N}, 1 \leq i \leq n, b_i = a\}$$

which is a nominal set. Indeed, it is stable under permutations of letters, and each word is finitely supported by the finite set of letters that appears in it.
Let us compute the minimal automaton for this language. Recall that for any nominal set A, $\Omega^A = \{P \subset A | P \text{ is finitely supported}\}$. The states object is $\mathbb{N}^*_{/\equiv_L}$ and here it is therefore the nominal set

$$\left\{u^{-1}L \subset \mathbb{N}^* \middle| u \in \mathbb{N}^*\right\} = \{L\} \cup \{\mathbb{N}^* a \mathbb{N}^* | a \in \mathbb{N}\} \cup \{\mathbb{N}^*\}$$

so that by Theorem 1, L is decomposition-regular.
To finally describe the minimal automaton, recall that the initial state, a fixed point, is simply the equivalence class L of ε, and an equivalence class is a final state if and only if it contains a language that contains the empty string, ε. The only such class is $\mathbb{N}^*$, therefore it is the only final state of the automaton. Then, the action $- \cdot a$ of a letter $a \in \mathbb{N}$ is given by $K \cdot a = a^{-1}K$. The following diagram sums up the construction, and the register automaton counterpart of this nominal automaton can be found in Francez and Kaminski [8, Figure 7]:



where the diamond state is the initial state and the double circle is a final state (in fact the only one in this case). Observe that states in the same column are in the same orbit. The orbit $\left\{a^{-1}L \middle| a \in \mathbb{N}\right\}$ can be thought of as a single state such that a transition from the initial state to this state-orbit writes the read letter (which is the first letter of the word) in a register. Reading the rest of the word, we loop on this state-orbit until we read a letter that is other than the one in the register. In that case we reach the final state on which we loop until the word is finished reading.

Let us compute the syntactic monoid of L. It is the set of functions $\left\{u^{-1}L \middle| u \in \mathbb{N}^*\right\} \to \left\{u^{-1}L \middle| u \in \mathbb{N}^*\right\}$ of the form $f_v : u^{-1}L \mapsto (uv)^{-1}L$ for some fixed v and with action $f_v \cdot \sigma = f_{v \cdot \sigma}$ because $v \mapsto f_v$ is a nominal monoid morphism $\mathbb{N}^* \twoheadrightarrow \mathrm{Syn}(L)$ so in particular an equivariant function. Then

$$\begin{aligned}\mathrm{Syn}(L) = \{f_\varepsilon\} \cup \{f_a | a \in \mathbb{N}\} \cup \left\{f_{abu} \middle| a \in \mathbb{N}, b \in \mathbb{N} \setminus \{a\}, u \in \mathbb{N} \setminus \{a\}^*\right\} \\ \cup \{f_{auav} | a \in \mathbb{N}, u, v \in \mathbb{N}^*\}\end{aligned}$$

and each member of this union is an orbit, so that $\mathrm{Syn}(L)$ is decomposition-finite.

In nominal automata theory, dK-finiteness is not an interesting notion:

Proposition 14. *The dK-finite objects in* $\mathbb{B}\,\mathrm{Aut}(\mathbb{N})$ *are exactly finite sets with the trivial action.*

Proof. The topos $\mathbb{B}\,\mathrm{Aut}(\mathbb{N})$ is a subtopos of $[\mathrm{Aut}(\mathbb{N})^{\mathrm{op}}, \mathrm{Set}]$, so dK-finite objects of $\mathbb{B}\,\mathrm{Aut}(\mathbb{N})$ have to be dK-finite objects of $[\mathrm{Aut}(\mathbb{N})^{\mathrm{op}}, \mathrm{Set}]$ (cf. Johnstone [7, Corollary 5.4.12]), which entails they have a finite underlying set according to Proposition 13. Because $\mathbb{B}\,\mathrm{Aut}(\mathbb{N})$ is atomic, we restrict to connected objects, i.e. non-empty transitive nominal sets, and show that the only transitive nominal set with finite underlying set is $\mathbb{1}$. According to Bojańczyk [3, Theorem 6.3], transitive nominal sets are of the form ${\mathbb{N}^{[F]}}_{/G}$ where F is a finite set, $\mathbb{N}^{[F]}$ is the set of injections from F to $\mathbb{N}$, and G a subgroup of $\mathfrak{S}_F$. But then the only case for which $\mathbb{N}^{[F]}_{/\mathfrak{S}_F}$ is a finite set is when $F \cong \emptyset$, and in that case ${\mathbb{N}^{[F]}}_{/G} \cong \mathbb{1}$.

6 Conclusion and Future Perspectives

Because the subobject classifier in Set played a crucial rôle in the Colcombet and Petrişan functorial viewpoint of automata, we adapted it to a wide class of toposes, recovering minimisation results, and adding Myhill-Nerode type theorems to it, as well as some discussions about the syntactic monoid of a language, everything internally to a given topos. The results still make sense for sets, and can be applied to the Myhill-Schanuel topos of nominal sets.

We would like to make more use of the enriched Colcombet and Petrişan functorial point of view of automata. For example, because everything is done using enriched category theory, a generalisation to non-cartesian monoidal closed categories is immediate, and should at least encompass the categories of Adámek, Milius and Urbat [2].

Acknowledgments. The author would like to thank Clemens Berger for the discussions and the help for the redaction of this article.

References

1. Acuña-Ortega, O., Linton, F.E.J.: Finiteness and decidability:I. In: Applications of Sheaves: Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra, and Analysis, Durham, July 9–21, 1977. Lecture Notes in Mathematics, vol. 753, pp. 80–100. Springer, Berlin, Heidelberg (1979). `https://doi.org/10.1007/BFb0061813`
2. Adámek, J., Milius, S., Urbat, H.: Syntactic Monoids in a Category. Proceedings CALCO'15 p. 1504.02694 (Jun 2015)
3. Bojańczyk, M.: Slightly Infinite Sets (Sep 2019), `https://www.mimuw.edu.pl/~bojan/paper/atom-book`
4. Bojańczyk, M., Klin, B., Lasota, S.: Automata theory in nominal sets. Logical Methods in Computer Science **10 issue 3**(4), 1402.0897 (Aug 2014). `https://doi.org/10.2168/LMCS-10(3:4)2014`
5. Carboni, A., Janelidze, G., Kelly, G.M., Paré, R.: On Localization and Stabilization for Factorization Systems. Applied Categorical Structures **5**(1), 1–58 (Mar 1997). `https://doi.org/10.1023/A:1008620404444`
6. Colcombet, T., Petrişan, D.: Automata Minimization: A Functorial Approach. Logical Methods in Computer Science ; Volume 16 p. 1712.07121 (Mar 2020). `https://doi.org/10.23638/LMCS-16(1:32)2020`
7. Johnstone, P.T.: Sketches of an Elephant. Oxford University Press (2002)
8. Kaminski, M., Francez, N.: Finite-memory automata. Theoretical Computer Science (1994)
9. Kelly, G.M.: Basic Concepts of Enriched Category Theory, London Mathematical Society Lecture Note Series, vol. 64. Cambridge University Press (1982), `http://www.tac.mta.ca/tac/reprints/articles/10/tr10abs.html`
10. Kleene, S.C.: Representation of Events in Nerve Nets and Finite Automata. Automata studies (34), 3–42 (1956)
11. Lucyshyn-Wright, R.B.B.: Enriched factorization systems. Theory and Applications of Categories **29**(18), 1401.0315 (Jan 2014)
12. Mac Lane, S., Moerdijk, I.: Sheaves in Geometry and Logic. Universitext, Springer (1994)
13. Moerdijk, I., Palmgren, E.: Wellfounded trees in categories. Annals of Pure and Applied Logic **104**(1), 189–218 (Jul 2000). `https://doi.org/10.1016/S0168-0072(00)00012-9`
14. Nerode, A.: Linear Automaton Transformations. Proceedings of the American Mathematical Society **9**(4), 541–544 (1958). `https://doi.org/10.2307/2033204`
15. Pin, J.É.: Mathematical Foundations of Automata Theory (Feb 2022), `https://www.irif.fr/~jep/PDF/MPRI/MPRI.pdf`

Appendix

Initial and terminal automata as enriched Kan extensions

Proof (Part of the missing proof of 4.). Let us compute the terminal automaton. Its states object is defined as the end

$$\int_{o\in\mathcal{O}_\Sigma} \mathcal{I}_\Sigma(\mathrm{st}, o) \pitchfork \underline{L}(o)$$

so by the end formula for right Kan extension, it is obtained by equalizing

$$\begin{tikzcd}
\prod_{o\in(\mathcal{O}_\Sigma)_0} \underline{L}(o)^{\mathcal{I}_\Sigma(\mathrm{st},o)} \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & \prod_{(o,o')\in(\mathcal{O}_\Sigma)_0^2} \underline{L}(o')^{\mathcal{I}_\Sigma(\mathrm{st},o)\times\mathcal{I}_\Sigma(o,o')}
\end{tikzcd}$$

for the φ and ψ of the lemma, which amounts to the equalizing of

$$\begin{tikzcd}
\Omega^{\Sigma^*} \arrow[r, shift left, "\mathrm{id}_{\Omega^{\Sigma^*}}"] \arrow[r, shift right, "\mathrm{id}_{\Omega^{\Sigma^*}}"'] & \Omega^{\Sigma^*}
\end{tikzcd}$$

because in each product, each other factor is $\mathbb{1}$ because either the exponent is $\emptyset$, either $\underline{L}(o) = \mathbb{1}$. Finally φ and ψ happens to be both the identity morphism.

Minimal automaton

Proof (Missing proof of 8.). We first show we have $(E_{\mathscr{I}}, M_{\mathscr{I}})$-factorizations. Consider two $\mathcal{V}$-functors F and G from $\mathscr{I}$ to $\mathcal{V}$ and α a $\mathcal{V}$-natural transformation from F to G which means that we have a collection $(\alpha_i : F(i) \to G(i))_i$ of arrows of $\mathcal{V}$ such that the diagram

$$\begin{tikzcd}
\mathscr{I}(i,j) \arrow[r, "F_{i,j}"] \arrow[d, "G_{i,j}"'] & \mathcal{V}(G(i),G(j)) \arrow[d, "\mathcal{V}(\alpha_i{,}G(j))"] \\
\mathcal{V}(F(i),F(j)) \arrow[r, "\mathcal{V}(F(i){,}\alpha_j)"'] & \mathcal{V}(F(i),G(j))
\end{tikzcd}$$

commutes for all couples (i,j) of objects of $\mathscr{I}$. Consider for any object i of $\mathscr{I}$ the (E,M)-factorization $F(i) \xrightarrow{\varepsilon_i} J(i) \xrightarrow{\mu_i} G(i)$ of α_i. In order to make the assignment $i \mapsto J(i)$ a $\mathcal{V}$-functor, we use the fact that the factorization system (E,M) is enriched by returning to the pullbacks definition; for all couples (i,j) of objects of $\mathscr{I}$ we have a solid diagram

$$\begin{tikzcd}
\mathscr{I}(i,j) \arrow[r, "G_{i,j}"] \arrow[d, "F_{i,j}"'] \arrow[dr, dotted, "J_{i,j}"] & \mathcal{V}(G(i),G(j)) \arrow[dr, "\mathcal{V}(\mu_i{,}G(j))"] & \\
\mathcal{V}(F(i),F(j)) \arrow[dr, "\mathcal{V}(F(i){,}\varepsilon_j)"'] & \mathcal{V}(J(i),J(j)) \arrow[r, "\mathcal{V}(J(i){,}\mu_j)"] \arrow[d, "\mathcal{V}(\varepsilon_i{,}J(j))"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{V}(J(i),G(j)) \arrow[d, "\mathcal{V}(\varepsilon_i{,}G(j))"] \\
& \mathcal{V}(F(i),J(j)) \arrow[r, "\mathcal{V}(F(i){,}\mu_j)"'] & \mathcal{V}(F(i),G(j))
\end{tikzcd}$$

where the outer hexagon commute because in may be identified with the commuting square above ($\alpha_i = \varepsilon_i \mu_i$ and $\alpha_j = \varepsilon_j \mu_j$). Since the factorization system on $\mathcal{V}$ is enriched, the front square of the diagram is a pullback so that we get a unique dotted arrow $J_{i,j}$, defining an action of the arrows of $\mathscr{I}$. The $\mathcal{V}$-functoriality of J is due to the $\mathcal{V}$-functoriality of F and G and the uniqueness of the action maps $J_{i,j}$. We can recognize on this same diagram the $\mathcal{V}$-naturality squares for μ (the top face of the cube) and ε (the left face), finally ensuring each arrow in $[\mathscr{I}, \mathcal{V}]$ admits an $(E_{\mathscr{I}}, M_{\mathscr{I}})$-factorization.

Now, $(E_{\mathscr{I}}, M_{\mathscr{I}})$ is indeed a factorization system. The factorization is unique because it is unique pointwise, and a $\mathcal{V}$-natural transformation is an isomorphism if and only if it is an isomorphism pointwise, so that $E_{\mathscr{I}}$ and $M_{\mathscr{I}}$ are closed under isomorphisms, and they are closed under composition too because so are E and M, and because composition of $\mathcal{V}$-natural transformations is also done pointwise.

Finally, we will make use of Proposition 7 to prove $(E_{\mathscr{I}}, M_{\mathscr{I}})$ is enriched. Consider a $\mathcal{V}$-natural transformation $\varepsilon : F \Rightarrow G$ in $[\mathscr{I}, \mathcal{V}]$ such that for each object i, ε_i is in E (i.e. such that ε is in $E_{\mathscr{I}}$). According to Proposition 7, because (E, M) is enriched, and because $v \otimes \varepsilon_i : v \otimes F(i) \to v \otimes G(i)$ is the copower of ε_i by v any object of $\mathcal{V}$, then $v \otimes \varepsilon_i$ is in E. It remains to show that $v \otimes \varepsilon \coloneqq (v \otimes \varepsilon_i)_i$ is the copowering of ε by v, so that by definition, $v \otimes \varepsilon$ is in $E_{\mathscr{I}}$, showing $E_{\mathscr{I}}$ is stable under copowers and thus by the Proposition, that $(E_{\mathscr{I}}, M_{\mathscr{I}})$ is enriched.

The $\mathcal{V}$-category $[\mathscr{I}, \mathcal{V}]$ where, recall, the object of arrows between F and G is the end $\int_{i:\mathscr{I}} \mathcal{V}(F(i), G(i))$, has all copowers that are given by tensoring the $\mathcal{V}$-functors pointwise. It is clear that tensoring pointwise gives a $\mathcal{V}$-functor because it boils down to postcomposing a $\mathcal{V}$-functor F by the $\mathcal{V}$-functor $v \otimes -$ with action on arrows between x and y the transpose of:

$$(v \otimes x) \otimes \mathcal{V}(x, y) \xrightarrow{\text{associator}} v \otimes (x \otimes \mathcal{V}(x, y)) \xrightarrow{v \otimes (\text{counit at } y)} v \otimes y.$$

This indeed defines a copowering in $[\mathscr{I}, \mathcal{V}]$:

$$\begin{aligned}
[\mathscr{I}, \mathcal{V}](v \otimes F, G) &= \int_{i:\mathscr{I}} \mathcal{V}(v \otimes F(i), G(i)) \\
&\cong \int_{i:\mathscr{I}} v \otimes \mathcal{V}(F(i), G(i)) \text{ by copowering in } \mathcal{V} \\
&\cong v \otimes \int_{i:\mathscr{I}} \mathcal{V}(F(i), G(i)) \text{ because continuous functors preserve ends} \\
&= v \otimes [\mathscr{I}, \mathcal{V}](F, G).
\end{aligned}$$