Optics, functorially

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CMCS 2024, Luxembourg

Motivation

- **Monads**: model notions of computations
  [Moggi'89,'91]

- **Monad transformers**: combine computational effects
  [Liang-Hudak-Jones'95, Moggi'97, Benton-Hughes-Moggi'00]

- **Lenses**: bidirectional (bx) transformations (**view-update** of databases)
  $(\mathsf{get} : A \longrightarrow B, \mathsf{put} : A \times B' \longrightarrow A')$ [Oles'82]

- **Lens transformers**?

- More generally, **(mixed) optics** as bx transformations

  Features: modularity, compositionality

  Applications: game theory, machine learning, database systems

  Therefore ... what about **optics transformers**?

Name	Description	Actions	Base
Adapter	$\mathbf{C}(S,A) \otimes \mathbf{D}(B,T)$	$(\mathbf{Optic}_{\mathrm{id},\mathrm{id}})$	$\mathcal{V}, \otimes$
Lens	$\mathbf{C}(S,A) \times \mathbf{D}(S \bullet B,T)$	$(\mathbf{Optic}_{\times,\bullet})$	$\mathcal{W}, \times$
Monoidal lens	$\mathbf{CCom}(S,A) \times \mathbf{C}(\mathcal{U}S \otimes B,T)$	$(\mathbf{Optic}_{\otimes,\mathcal{U}\times})$	$\mathcal{W}, \times$
Algebraic lens	$\mathbf{C}(S,A) \times \mathbf{D}(\Psi S \bullet B,T)$	$(\mathbf{Optic}_{\mathcal{U}\times,\mathcal{U}\bullet})$	$\mathcal{W}, \times$
Monadic lens	$\mathcal{W}(S,A) \times \mathcal{W}(S \times B,\Psi T)$	$(\mathbf{Optic}_{\times,\rtimes})$	$\mathcal{W}, \times$
Linear lens	$\mathbf{C}(S,[B,T] \bullet A)$	$(\mathbf{Optic}_{\bullet,\otimes})$	$\mathcal{V}, \otimes$
Prism	$\mathbf{C}(S,T \bullet A) \times \mathbf{D}(B,T)$	$(\mathbf{Optic}_{\bullet,+})$	$\mathcal{W}, \times$
Coalg. prism	$\mathbf{C}(S,\Theta T \bullet A) \times \mathbf{D}(B,T)$	$(\mathbf{Optic}_{\mathcal{U}\bullet,\mathcal{U}+})$	$\mathcal{W}, \times$
Grate	$\mathbf{D}([S,A] \bullet B,T)$	$(\mathbf{Optic}_{\{\ ,\ \},\bullet})$	$\mathcal{V}, \otimes$
Glass	$\mathbf{C}(S \times [[S,A],B],T)$	$(\mathbf{Optic}_{\times[\ ,\ ],\times[\ ,\ ]})$	$\mathcal{W}, \times$
Affine traversal	$\mathbf{C}(S,T + A \otimes \{B,T\})$	$(\mathbf{Optic}_{+\otimes,+\otimes})$	$\mathcal{W}, \times$
Traversal	$\mathcal{V}(S,\sum^n A^n \otimes [B^n,T])$	$(\mathbf{Optic}_{\mathrm{Pw},\mathrm{Pw}})$	$\mathcal{V}, \otimes$
Kaleidoscope	$\sum_n \mathcal{V}([A^n,B],[S^n,T])$	$(\mathbf{Optic}_{\mathrm{App},\mathrm{App}})$	$\mathcal{V}, \otimes$
Setter	$\mathcal{V}([A,B],[S,T])$	$(\mathbf{Optic}_{\mathrm{ev},\mathrm{ev}})$	$\mathcal{V}, \otimes$
Fold	$\mathcal{V}(S,\mathcal{L}A)$	$(\mathbf{Optic}_{\mathrm{Foldable},*})$	$\mathcal{V}, \otimes$

Table of optics [Clarke et al. '24]

Monoidal categories

Monoidal category: a category $\mathscr{M}$ equipped with a tensor product (bifunctor)

$$\mathscr{M} \times \mathscr{M} \longrightarrow \mathscr{M} \ , \ (M, N) \mapsto M \otimes N$$

and a unit object I, such that $\otimes$ is associative and unital up to coherent isomorphism

Examples

- Any category $\mathscr{M}$ with finite (co)products
- Endofunctors $[\mathscr{A}, \mathscr{A}]$, with functor composition
- Presheaves $[\mathscr{A}^{\mathsf{op}}, \mathsf{Set}]$ over a (small) category $\mathscr{A}$, with Day convolution
- The Eilenberg-Moore category of algebras of a commutative monad on a monoidal category

Actegories

[Beńabou'67, McCrudden'00, Capucci-Gavranović'22]

$(\mathscr{M}, \otimes, I)$ monoidal category

$\mathscr{M}$**-actegory**: a category $\mathscr{A}$ equipped with an action (bifunctor)

$$\mathscr{M} \times \mathscr{A} \longrightarrow \mathscr{A} \;,\; (M, A) \mapsto M \cdot A$$

associative and unital up to coherent isomorphism

$$(M \otimes N) \cdot A \cong M \cdot (N \cdot A) \;,\; I \cdot A \cong A$$

Examples

- Any monoidal category $\mathscr{M}$ acts on itself via the tensor product
- For any category $\mathscr{A}$, $[\mathscr{A}, \mathscr{A}]$ acts on $\mathscr{A}$ via functor application
- The Kleisli category of a strong monad on a monoidal category

Actegories

Lax $\mathscr{M}$-morphism: functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ between $\mathscr{M}$-actegories, endowed with natural transformation $\mathrm{st} : M \cdot FA \longrightarrow F(M \cdot A)$ (**strength**), compatible with the $\mathscr{M}$-actions

Example. If $\mathscr{M}$ is a cartesian category acting on itself, then an $\mathscr{M}$-lax endofunctor $\mathscr{M} \longrightarrow \mathscr{M}$ is precisely a **strong** functor.

Remark. A **lax $\mathscr{M}$-morphism structure** on an endofunctor F on an $\mathscr{M}$-actegory is the same as a **lifting** of the $\mathscr{M}$-action to $\mathrm{Coalg}(F)$, such that the forgetful functor becomes strict $\mathscr{M}$-morphism (strength is identity)

Colax $\mathscr{M}$-morphism: lax $\mathscr{M}$-morphism between opposite actegories. A colax $\mathscr{M}$-morphism F comes equipped with a **costrength** $\mathrm{cst} : F(M \cdot A) \longrightarrow M \cdot FA$

Example. Let $\mathscr{M}$ be a cartesian category acting on itself. An endofunctor on $\mathscr{M}$ is a colax $\mathscr{M}$-morphism (also known as **costrong** functor) iff it is **copointed** [B-Pantelimon'24]

(Co)Lax $\mathscr{M}$-transformation: natural transformation between (co)lax $\mathscr{M}$-morphisms, compatible with their (co)strengths

The Para construction

[Wood'76, Hermida-Tennent'12, Fong-Spivak-Tuyéras'19, Capucci-Gavranović-Hedges'20]

Let $\mathscr{A}$ be an $\mathscr{M}$-actegory

Para$(\mathscr{A})$: bicategory which **"adds parameters"** to $\mathscr{A}$

- the objects are those of $\mathscr{A}$
- the morphisms are M-parametrised morphisms $(M \in \mathscr{M}, f : M \cdot A \longrightarrow B)$
- 2-cells are given by **reparametrisation**

Remarks

- The construction $\mathscr{M}$-$\mathbf{Act}_l \longrightarrow \mathbf{Bicat}$, $\mathscr{A} \mapsto \mathbf{Para}(\mathscr{A})$ is functorial with respect to **lax $\mathscr{M}$-morphisms**.
- There is a 2-opfibration $\mathbf{Para}(\mathscr{A}) \longrightarrow \mathbf{B}\mathscr{M}$ over the delooping of $\mathscr{M}$, by projecting parameters.
- Dually, $\mathbf{CoPara}(\mathscr{A}) = \mathbf{Para}(\mathscr{A}^{\mathrm{op}})$ gives a functor $\mathscr{M}$-$\mathbf{Act}_c \longrightarrow \mathbf{Bicat}$ and a 2-fibration over $\mathbf{B}\mathscr{M}$

Optics

Informally: optics are coupled pairs of coparametrised and parametrised morphisms, but with externally **unobservable** joint parameter

Let $\mathscr{A}, \mathscr{B}$ be two $\mathscr{M}$-actegories

A **(mixed) optic** from (A, B) with the **focus** on (A', B') is an element of the coend

$$\mathbf{Optic}_{\mathscr{A},\mathscr{B}}((A, B), (A', B')) = \int^{M} \mathscr{A}(A, M \cdot A') \times \mathscr{B}(M \cdot B', B)$$

where $A, A' \in \mathscr{A}$ and $B, B' \in \mathscr{B}$

Optics are arrows of a **category** $\mathbf{Optic}_{\mathscr{A},\mathscr{B}}$, which comes with an identity on objects fully faithful functor

$$\mathscr{A}^{\mathsf{op}} \times \mathscr{B} \longrightarrow \mathbf{Optic}_{\mathscr{A},\mathscr{B}}$$

Lenses arise when $\mathscr{M} = \mathscr{A} = \mathscr{B}$ and both the monoidal structure and the actions are given by the cartesian product

$$\int^{M} \mathscr{M}(A, M \times A') \times \mathscr{M}(M \times B', B) \quad \cong \quad \mathscr{M}(A, A') \times \mathscr{M}(A \times B', B)$$

Optics, functorially

Optic$_{\mathscr{A},\mathscr{B}}$ as a **bicategory**:
[Braithwaite et al.'21]

$$\begin{array}{ccc} \mathbf{Optic}_{\mathscr{A},\mathscr{B}} & \longrightarrow & \mathbf{Para}(\mathscr{B}) \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathbf{CoPara}(\mathscr{A}) & \longrightarrow & \mathbf{B}\mathscr{M} \end{array}$$

The 1-cells in the bicategory **Optic**$_{\mathscr{A},\mathscr{B}}$ are pairs of coparametrised, respectively parametrised morphisms

$$(M, f : A \longrightarrow M \cdot A', g : M \cdot B' \longrightarrow B)$$

explicitly keeping track of the residual, without constraints (which now live in the 2-cells of **Optic**$_{\mathscr{A},\mathscr{B}}$)

The functors $\mathbf{Para} : \mathscr{M}\text{-}\mathbf{Act}_l \longrightarrow \mathbf{Bicat}$ and $\mathbf{CoPara} : \mathscr{M}\text{-}\mathbf{Act}_c \longrightarrow \mathbf{Bicat}$ consequently induce a pseudofunctor [B-Pantelimon'24]

$$\mathbf{Optic} : \mathscr{M}\text{-}\mathbf{Act}_c \otimes \mathscr{M}\text{-}\mathbf{Act}_l \longrightarrow \mathbf{Bicat}$$

Optics versus lenses, functorially

Let $\mathscr{M} = \mathscr{A} = \mathscr{B}$, with monoidal structure and $\mathscr{M}$-actions given by the cartesian product

There is a **local adjunction** between the bicategory of optics and the (discrete) bicategory of lenses [Gavranović '22]

$$\mathbf{Lens}_{\mathscr{M}}((A, A'), (B, B')) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Optic}_{\mathscr{M},\mathscr{M}}((A, A'), (B, B'))$$

which becomes a **bijection** of homsets when restricted to the 1-category of optics.

Let $F, G : \mathscr{M} \longrightarrow \mathscr{M}$ be costrong, respectively strong functors

Then there is a **morphism of adjunctions** given by $\mathbf{Lens}(F, G)$ and $\mathbf{Optic}(F, G)$ [B-Pantelimon'24]

$$\begin{array}{ccc}
\mathbf{Lens}_{\mathscr{M}}((A, A'), (B, B')) & \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} & \mathbf{Optic}_{\mathscr{M},\mathscr{M}}((A, A'), (B, B')) \\
{\scriptstyle \mathbf{Lens}(F,G)} \downarrow & & \downarrow {\scriptstyle \mathbf{Optic}(F,G)} \\
\mathbf{Lens}_{\mathscr{M}}((FA, GA'), (FB, GB')) & \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} & \mathbf{Optic}_{\mathscr{M},\mathscr{M}}((FA, GA'), (FB, GB'))
\end{array}$$

Optics, 2-functorially

Dependent optics [Vertechi'22] generalise (mixed) optics, replacing the actions of the monoidal category $\mathscr{M}$ by a pair of pseudofunctors

$$\mathbf{B}^{\mathsf{op}} \longrightarrow \mathbf{Cat}$$

where $\mathbf{B}$ is a bicategory (taking $\mathbf{B}$ to be the delooping of $\mathscr{M}$ recovers usual optics)

Our previous **Optic** construction **extends to dependent optics** when lax and colax $\mathscr{M}$-morphisms are replaced by lax and colax natural transformations.

Concluding: open problems

- Better understand optics properties deriving from their inherent fibrational nature
- Instantiate $\textbf{Optic}(F, G)$ to various classes of optics (prisms, traversables, etc.); gain more intuition on these
- Find (if any) connections with monad-comonad interaction laws

Thank you for your attention!