Higher Coalgebra: A Homotopy Theory Of Behaviour

Workshop on Coalgebraic Methods in Computer Science

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Outline

Introduction and Motivation

Motivation

Homotopy theory and algebraic topology for behaviour

- (Weak) homotopy equivalence of systems
- Homotopy and (co)homology to find behavioural obstructions
- Homotopy-invariant logic

Examples

- Concurrent computing — detecting deadlocks[1]
- Distributed computing — computability results[2]
- Hybrid computing — detecting and handling Zeno behaviour[3]
- Modal logic for higher dimensional automata[4]

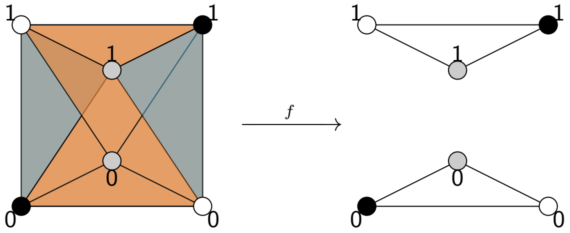
[1] Lisbeth Fajstrup et al. *Directed Algebraic Topology and Concurrency*. Springer, 2016, p. 167. 1 p. ISBN: ISBN 978-3-319-15397-1. DOI: 10.1007/978-3-319-15398-8.

[2] Maurice Herlihy, Dmitry Kozlov, and Sergio Rajsbaum. *Distributed Computing Through Combinatorial Topology*. 1st ed. San Francisco, CA, USA: Morgan Kaufmann Publishers Inc., Nov. 2013. 336 pp. ISBN: 978-0-12-404578-1.

[3] Aaron D. Ames and Shankar Sastry. "Characterization of Zeno Behavior in Hybrid Systems Using Homological Methods". In: *Proceedings of the 2005, American Control Conference, 2005*. ACC 2005. June 2005, 1160–1165 vol. 2. DOI: 10.1109/ACC.2005.1470118.

[4] Cristian Prisacariu. *Higher Dimensional Modal Logic*. 2014. arXiv: 1405.4100. URL: http://arxiv.org/abs/1405.4100. preprint.

Obstructions in Asynchronous Systems



Computation as mapping problem

- Simplicial complexes to model input and output of problem
- Binary consensus solvable if and only if simplicial map f exists
- Solution is obstructed because input space is path connected but output is not[5]
- However: static model of behaviour
- Difficult to adapt to other computational features (e.g. memory interaction)

[5] Maurice Herlihy and Nir Shavit. "The Topological Structure of Asynchronous Computability". In: *J. ACM* 46.6 (Nov. 1, 1999), pp. 858–923. ISSN: 0004-5411. DOI: 10.1145/331524.331529.

Behaviour via Coalgebras

- Behaviour from repeated observation of a space X via map $c\colon X \to FX$
- Functor $F\colon \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$ determines the type of observations

Example (Hybrid Systems as Coalgebras)

- Hybrid system as space and a coalgebra that specifies the trajectories in the space[6]
- **Top** "convenient" category of topological spaces that is (co)complete, Cartesian closed, and has CW-complexes, like compactly generated Hausdorff spaces or Δ-spaces[7]
- Define a functor $H\colon \mathbf{Top} \to \mathbf{Top}$ by

$$HX = \{(\varrho, d) \in X^{\mathbb{R}_{\geq 0}} \times [0, \infty] \mid \varrho \circ \min(-, d) = \varrho\} \qquad \text{and} \qquad (Hf)(\varrho, d) = (f \circ \varrho, d)$$

- A coalgebra $c\colon X \to HX$ continuously assigns to $x \in X$ a pair (ϱ, d) of trajectory $\varrho\colon \mathbb{R}_{\geq 0} \to X$ that is constant after duration d.
- Can be refined to ensure that the trajectory $c(x)$ has x as starting point etc.

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[6] Renato Neves et al. "Continuity as a Computational Effect". In: *JLAMP*. Articles Dedicated to Prof. J. N. Oliveira on the Occasion of His 60th Birthday 85 (5, Part 2 Aug. 1, 2016), pp. 1057–1085. ISSN: 2352-2208. DOI: 10.1016/j.jlamp.2016.05.005.

[7] J. Peter May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, Sept. 1999. 254 pp. ISBN: 978-0-226-51183-2. URL: https://www.math.uchicago.edu/~may/CONCISE/.

Behaviour of Coalgebras

- Behaviour of coalgebra c by recursively expanding observations into a sequence

$$X \xrightarrow{c} FX \xrightarrow{Fc} F(FX) \xrightarrow{F(Fc)} \cdots$$

- Gives in the limit a total view on behaviour of c, if that exists[8]
- Traces and logical formulas are partial view on this sequence
- Coalgebra homomorphisms relate the behaviour of systems

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "c"'] & Y \arrow[d, "d"] \\ FX \arrow[r, "Ff"] & FY \end{tikzcd}$$

- Coalgebra homomorphisms preserve and reflect the behaviour
- Behaviour of the image f in d is equal to that of c
- Often coincide with bisimilarity[9], but we want homotopic behaviour

[8] Michael Barr. "Terminal Coalgebras in Well-Founded Set Theory". In: *TCS* 114.2 (1993), pp. 299–315. DOI: 10.1016/0304-3975(93)90076-6.

[9] Sam Staton. "Relating Coalgebraic Notions of Bisimulation". In: *LMCS* 7.1 (2011), pp. 1–21. DOI: 10.2168/LMCS-7(1:13)2011.

Higher Coalgebra in Topological Models

Homotopy Theory via Topological Enrichment

Topological Enrichment

$\underline{\mathcal{C}}$ is a $\mathbf{Top}$-enriched category if

- it has objects
- it has a space $\underline{\mathcal{C}}(X, Y) \in \mathbf{Top}$ for all objects X, Y
- there are continuous composition maps $c_{X,Y,Z} \colon \underline{\mathcal{C}}(Y, Z) \times \underline{\mathcal{C}}(X, Y) \to \underline{\mathcal{C}}(X, Z)$
- there is an identity $\mathrm{id}_X \colon * \to \underline{\mathcal{C}}(X, X)$ for all objects X
- an associativity and two unit diagrams commute

Enrichment (plus other things) enables homotopy theory[10]

- Define a homotopy $h \colon f \Rightarrow g$ between $f, g \in \underline{\mathcal{C}}(X, Y)$ to be a continuous map $h \colon [0, 1] \to \underline{\mathcal{C}}(X, Y)$ with $h(0) = f$ and $h(1) = g$
- Write $f \sim g$ if there is some homotopy $f \Rightarrow g$

[10] Emily Riehl. *Categorical Homotopy Theory*. New Mathematical Monographs 24. Cambridge University Press, 2014. ISBN: 978-1-107-04845-4. URL: https://math.jhu.edu/~eriehl/cathtpy/; Michael Shulman. *Homotopy Limits and Colimits and Enriched Homotopy Theory*. June 30, 2009. DOI: 10.48550/arXiv.math/0610194. arXiv: math/0610194. preprint.

Behaviour up to Homotopy

Example

- Continuous maps form a space $\underline{\mathbf{Top}}(X, Y)$ and composition is continuous
- This makes $\underline{\mathbf{Top}}$ a $\mathbf{Top}$-enriched category
- Call $f \colon X \to Y$ a homotopical coalgebra morphism from $c \colon X \to HX$ to $d \colon Y \to HY$ if it comes with a homotopy $h \colon Hf \circ c \Rightarrow d \circ f$
- The functor H is $\mathbf{Top}$-enriched, that is, $H_{X,Y} : \mathbf{Top}(X, Y) \to \mathbf{Top}(HX, HY)$ is continuous
- Hence, homotopy $h : f \to g$ can be mapped to a homotopy $Hh \colon Hf \to Hg$ by $Hh = H_{X,Y} \circ h$
- Obtain a sequence of homotopies

$$\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "f"'] & HX \arrow[r, "Hc"] \arrow[d, dashed, "Hf"] \arrow[dl, phantom, "\Leftarrow h"] & H(HX) \arrow[r, "H(Hc)"] \arrow[d, dashed, "H(Hf)"] \arrow[dl, phantom, "\Leftarrow Hh"] & H^3X \arrow[r] \arrow[d, "H^3 f"] \arrow[dl, phantom, "\Leftarrow H(Hh)"] & \cdots \\
Y \arrow[r, "d"'] & HY \arrow[r, "Hd"'] & H(HY) \arrow[r, "H(Hd)"'] & H^3Y \arrow[r] & \cdots
\end{tikzcd}$$

Higher Coalgebra

Commutativity up to homotopy

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "f"'] \arrow[dr, phantom, "\sim"] & FX \arrow[d, "Ff"] \\ Y \arrow[r, "d"'] & FY \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "c"] \arrow[d, "f"] \arrow[dr, phantom, "\sim"] & FX \arrow[r, "Fc"] \arrow[d, "Ff"] \arrow[dr, phantom, "\sim"] & F(FX) \arrow[r, "F(Fc)"] \arrow[d, "F(Ff)"] \arrow[dr, phantom, "\sim"] & \cdots \\ Y \arrow[r, "d"] & FY \arrow[r, "Fd"] & F(FY) \arrow[r, "F(Fd)"] & \cdots \end{tikzcd}$$

Long-term: homotopy theory of systems as higher coalgebra theory

- inspired by coalgebra[11] and higher algebra[12]
- use $(\infty, 1)$-categories to track homotopies
- the homotopy coherent nerve $N\underline{\mathcal{C}}$ of a $\mathbf{Top}$-enriched category is a model of $(\infty, 1)$-categories[13]
- Directions: coalgebra in higher categories, obstruction theory via (co)homology, homotopy-invariant modal logic

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[11] Jan Rutten. "Universal Coalgebra: A Theory of Systems". In: *TCS* 249.1 (2000), pp. 3–80. ISSN: 0304-3975. DOI: 10.1016/S0304-3975(00)00056-6.

[12] Jacob Lurie. *Higher Algebra*. Sept. 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[13] Jacob Lurie. *Higher Topos Theory*. Annals of Mathematics Studies 170. Princeton University Press, 2009. ISBN: 978-0-691-14049-0. arXiv: math/0608040.

Coalgebra in Higher Categories

Formal Coalgebra in 2-Categories

- Work in 2-category $\mathcal{C}$: $\mathbf{Cat}$, $\mathcal{V}$-$\mathbf{Cat}$, $\mathbf{Fib}$, $\mathbf{qCat}_2$ (homotopy 2-category of quasi-categories)[14], $\mathfrak{h}\mathcal{K}$ (homotopy 2-category of ∞-cosmos $\mathcal{K}$)[15]
- Define coalgebra objects (special 2-limits, inserters[16])
- Define 2-category $\mathcal{C}^{\circlearrowleft}$ of endomorphisms, distributive laws and distributive law morphisms with forgetful 2-functor $U \colon \mathcal{C}^{\circlearrowleft} \to \mathcal{C}$

$$
\begin{tikzcd}
A \arrow[d, "f"'] \\
A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
A \arrow[r, "k"'] \arrow[ur, phantom, "\Nearrow\delta"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, bend left, "k"] \arrow[r, bend right, "k'"'] \arrow[r, phantom, "\Downarrow\alpha"] & B
\end{tikzcd}
$$

Theorem

If the 2-category $\mathcal{C}$ has a choice of coalgebra objects for all endomorphisms, then there is a product-preserving 2-functor $\mathrm{CoAlg} \colon \mathcal{C}^{\circlearrowleft} \to \mathcal{C}$ with a 2-natural transformation $p \colon \mathrm{CoAlg} \to U$.

[14] Emily Riehl. *Categorical Homotopy Theory*. New Mathematical Monographs 24. Cambridge University Press, 2014. ISBN: 978-1-107-04845-4. URL: https://math.jhu.edu/~eriehl/cathtpy/.

[15] Emily Riehl and Dominic Verity. *Elements of ∞-Category Theory*. Cambridge University Press (CUP), 2022. ISBN: 978-1-108-93688-0. DOI: 10.1017/9781108936880.

[16] Claudio Hermida and Bart Jacobs. "Structural Induction and Coinduction in a Fibrational Setting". In: *Information and Computation* 145 (1997), pp. 107–152. DOI: 10.1006/inco.1998.2725.

What's the point?

Many known results are instances of 2-functoriality

- transport of adjunctions
- monoidal structure on coalgebras
- determinisation
- soundness of coalgebraic modal logic

For an appropriate 2-categorical definition of colimit we get a known result in general:

Theorem

If $\mathcal{C}$ is Cartesian closed, then $p\colon \mathrm{CoAlg} \to U$ creates colimits.

Instance: homotopy colimits in quasi-categories

Direction 1

Develop coalgebra further in higher categories, including enriched for good computation methods

Behavioural Obstructions in Topological Models

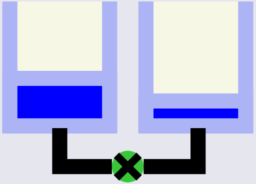
Zeno Behaviour

Sisyphus pumps water

- Two water tanks connected by a pump
- Pumps water until tank is empty and then switches direction
- Two states for the pumping directions
- Guards enable transitions
- Two sets of differential equations for linear flow

$$\begin{array}{ll} \dot{x}_1 = 1 & \dot{x}_1 = -1 \\ \dot{x}_2 = -1 & \dot{x}_2 = 1 \end{array}$$

$$\begin{tikzcd} s_1 \arrow[rr, bend left, "x_2 = 0"] & & s_2 \arrow[ll, bend left, "x_1 = 0"] \end{tikzcd}$$



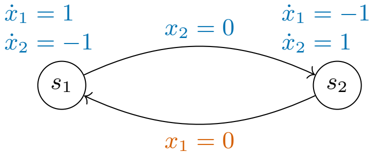
Not physically realisable

Infinite switching in finite time when both tanks are empty

Modelling the Water Tanks

Domains and guards

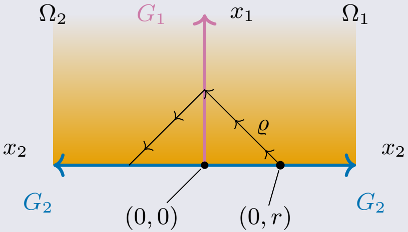
$$
\begin{aligned}
\Omega_k &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_k \geq 0\}, k \in \{1, 2\} \\
G_1 &= \{(x_1, x_2) \in \Omega_1 \mid x_2 = 0\} \\
G_2 &= \{(x_1, x_2) \in \Omega_2 \mid x_1 = 0\}
\end{aligned}
$$



Hybrid computation as coalgebra on colimit space

$$
D = \begin{tikzcd}
 & G_1 \arrow[r] \arrow[dr] & \Omega_1 \\
\{(0,0)\} \arrow[ur] \arrow[dr] & & \\
 & G_2 \arrow[r] \arrow[ur] & \Omega_2
\end{tikzcd}
$$

$$
\begin{aligned}
S_1 &= \operatorname{colim} D \\
c_1 &: S_1 \to HS_1
\end{aligned}
$$



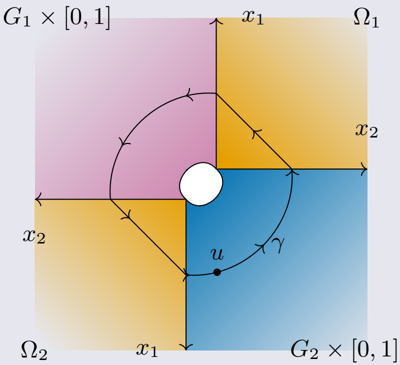
Realisable Sisyphus

Switching

- Switching takes time
- But it is irrelevant how much
- Trajectories in homotopy colimit $\operatorname{hcolim} D$ of D!

$$S_2 = \operatorname{hcolim} D$$
$$c_2 \colon S_2 \to HS_2$$

$$\Omega_k = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_k \geq 0\}$$
$$G_1 = \{(x_1, 0) \in \mathbb{R}^2 \mid x_1 \geq 1\}$$
$$G_2 = \{(0, x_2) \in \mathbb{R}^2 \mid x_2 \geq 1\}$$



Postulate

Any physically realisable model must have a coalgebra map up to homotopy into c_2.

Homotopical Obstruction to Realisability

Water tank pump not realisable

- Let $f\colon S_1 \to S_2$ be a map with a homotopy $h\colon c_2 \circ f \Rightarrow Hf \circ c_1$ (endpoint-preserving)
- This allows us to show that any loop in S_2 can be contracted to a constant path
- But there is a hole in S_2!
- Thus such h cannot exist and c_1 is not realisable

Dual use

The other way around: c_2 forces system to be realisable[17]

Direction 2

Systematic development of tools to detect obstructions, like (co)homology.

[17] Tyler Westenbroek et al. "Smooth Approximations for Hybrid Optimal Control Problems with Application to Robotic Walking". In: *IFAC-PapersOnLine*. 7th IFAC Conference on Analysis and Design of Hybrid Systems ADHS 2021 54.5 (Jan. 1, 2021), pp. 181–186. ISSN: 2405-8963. DOI: 10.1016/j.ifacol.2021.08.495.

Homotopy-Invariant Modal Logic

Modal Logic on HDA

Show modalities and homotopy axiom[18]

$$\varphi ::= p \mid \bot \mid \varphi \to \varphi \mid \Diamond^{\uparrow}\varphi \mid \Diamond^{\downarrow}\varphi$$

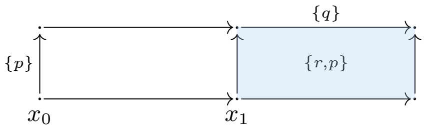
- $\Diamond^{\uparrow}\varphi$ holds if some action can be started and φ holds during execution
- $\Diamond^{\downarrow}\varphi$ holds if some action can be ended and φ holds afterwards

Interpretation over an HDA with cubes X

$$\begin{aligned}
[\![\Diamond^{\uparrow}\varphi]\!]_n &= \{x \in X_n \mid \exists x' \in X_{n+1}.\, x \text{ is a boundary cell of } x' \text{ and } x' \in [\![\varphi]\!]_{n+1}\} \\
[\![\Diamond^{\downarrow}\varphi]\!]_{n+1} &= \{x \in X_{n+1} \mid \exists x' \in X_n.\, x' \text{ is a boundary cell of } x \text{ and } x' \in [\![\varphi]\!]_n\} \\
x \vDash \varphi &\iff \exists n.\, x \in [\![\varphi]\!]_n
\end{aligned}$$

[18] Cristian Prisacariu. "Modal Logic over Higher Dimensional Automata". In: *Proc. of CONCUR 2010*. 2010, pp. 494–508. DOI: 10.1007/978-3-642-15375-4_34.

Homotopy-Invariance for HDA Logic



Example

$$x_0 \vDash \lozenge^{\uparrow} p \qquad x_1 \vDash \lozenge^{\uparrow}\lozenge^{\uparrow}\lozenge^{\downarrow} q$$

$$x_1 \vDash \lozenge^{\uparrow}\lozenge^{\uparrow} r \wedge p \qquad x_1 \vDash \lozenge^{\uparrow}\lozenge^{\downarrow}\lozenge^{\uparrow} q$$

Interchange Axioms[19]

$$\lozenge^{\uparrow}\lozenge^{\uparrow}\lozenge^{\downarrow}\varphi \to \lozenge^{\uparrow}\lozenge^{\downarrow}\lozenge^{\uparrow}\varphi \tag{A10}$$

$$\lozenge^{\uparrow}\lozenge^{\downarrow}\lozenge^{\downarrow}\varphi \to \lozenge^{\downarrow}\lozenge^{\uparrow}\lozenge^{\downarrow}\varphi \tag{A10'}$$

[19] Cristian Prisacariu. *Higher Dimensional Modal Logic*. 2014. arXiv: 1405.4100. URL: http://arxiv.org/abs/1405.4100. preprint.

Coalgebraic Modal Logic

One view based on dual adjunctions, so-called logical connections[20]

$$\begin{tikzcd} \mathcal{C} \arrow[loop left, "F"] \arrow[rr, bend left, "P"] \arrow[rr, phantom, "\perp"] & & \mathcal{D}^{\mathrm{op}} \arrow[ll, bend left, "Q"] \arrow[loop right, "L^{\mathrm{op}}"] \end{tikzcd} \quad \text{and} \quad \varrho \colon PF \to L^{\mathrm{op}} P \quad \text{and} \quad \alpha \colon L\Phi \to \Phi$$

Components

- $\mathcal{C}$ category for "states" in coalgebras
- F behaviour functor to get coalgebras $X \to FX$
- $\mathcal{D}$ typically category of algebras for logical operators
- L specifies modal operators
- initial algebra α for syntax
- distributive law $\varrho \colon LP \to PF$ to give semantics of formulas in a coalgebra
- $P \dashv Q$ is often concrete duality by mapping into dualising object

[20] Dusko Pavlovic, Michael W. Mislove, and James Worrell. "Testing Semantics: Connecting Processes and Process Logics". In: *Proceedings of Algebraic Methodology and Software Technology, 11th International Conference, AMAST 2006*. Ed. by Michael Johnson and Varmo Vene. Vol. 4019. Lecture Notes in Computer Science. Springer, 2006, pp. 308–322. DOI: 10.1007/11784180_24; Toby Wilkinson. "Enriched Coalgebraic Modal Logic". PhD thesis. 2013. URL: http://eprints.soton.ac.uk/354112/.

Modal Logic for General Coinductive Predicates

Previous picture is restricted to logic for behavioural equivalence/bisimilarity!

$$
\begin{tikzcd}
\mathcal{E} \arrow[loop left, "\overline{F}"] \arrow[dd, "p"] \arrow[dr, bend left, "\overline{P}"] \arrow[dr, phantom, "\perp"] & \\
 & \mathcal{D}^{\mathrm{op}} \arrow[ul, bend left, "\overline{Q}"] \arrow[dl, bend left, "Q"] \arrow[loop right, "L^{\mathrm{op}}"] \\
\mathcal{B} \arrow[loop left, "F"] \arrow[ur, bend left, "P"] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}
$$

Components[21]

- $p \colon \mathcal{E} \to \mathcal{B}$ fibration
- coalgebras for $\overline{F}$ are proofs of coinductive predicates
- final coalgebras in fibres are typically called coinductive predicates
- soundness (adequacy) and completeness (expressiveness) results provable in this setting

[21] Clemens Kupke and Jurriaan Rot. "Expressive Logics for Coinductive Predicates". In: *Logical Methods in Computer Science* Volume 17, Issue 4 (Dec. 15, 2021). DOI: 10.46298/lmcs-17(4:19)2021.

Homotopy-Invariance for Coinductive Predicates of Hybrid Systems I/III

Focus on the fibration side

Closed predicates

$$\mathbf{cPred} = \begin{cases} \text{objects:} & (X, P) \text{ with } X \in \mathbf{Top}, P \subseteq X \text{ closed} \\ \text{morphisms:} & (X, P) \to (Y, Q) \text{ continuous map } f \colon X \to Y \text{ with } f^{\to}(P) \subseteq Q \end{cases}$$

- Projection $p \colon \mathbf{cPred} \to \mathbf{Top}$ is fibration
- Reindexing by taking preimages: for $f \colon X \to Y$ continuous, define $f^* \colon \mathbf{cPred}_Y \to \mathbf{cPred}_X$ by

$$f^*(Y, Q) = (X, f^{\leftarrow}(Q))$$

- Fibration $p \colon \mathbf{cPred} \to \mathbf{Top}$ is $\mathbf{Top}$-enriched where $\mathbf{cPred}((X, P), (Y, Q))$ has subspace topology

Enriched Fibrations

Several choices (I know)

1. (large) fibration over the same base $\mathcal{B}$ as $\mathcal{V} \to \mathcal{B}$ and fibred enrichment (Shulman[22])
2. base and total category enriched over a fixed $\mathcal{V}$ (Wong and Beardsley[23])
3. internal fibration in $\mathcal{V}$-Cat (equivalent to previous, Wong[24])
4. enriched in $\mathcal{V}$ and fibration on underlying category (B.)
5. enriched over other fibration, such that hom-objects and composition agree (Vasilakopoulou[25]); can be reformulated as generalisation of Shulman's version

[22] Michael Shulman. "Enriched Indexed Categories". In: *TAC* 28.21 (2013), pp. 616–695. URL: http://www.tac.mta.ca/tac/volumes/28/21/28-21abs.html.

[23] Jonathan Beardsley and Liang Ze Wong. "The Enriched Grothendieck Construction". In: *Advances in Mathematics* 344 (Feb. 2019), pp. 234–261. ISSN: 00018708. DOI: 10.1016/j.aim.2018.12.009. arXiv: 1804.03829 [math].

[24] Liang Ze Wong. "The Grothendieck Construction in Enriched, Internal and ∞-Category Theory". Thesis. 2019. URL: https://digital.lib.washington.edu:443/researchworks/handle/1773/44365.

[25] Christina Vasilakopoulou. *On Enriched Fibrations*. July 6, 2018. DOI: 10.48550/arXiv.1801.01386. arXiv: 1801.01386. preprint.

Homotopy-Invariance for Coinductive Predicates of Hybrid Systems II/III

Reindexing along a homotopy

- For homotopy $h\colon f \Rightarrow g\colon X \to Y$, define

$$h^*(Y, Q) = \{x \in X \mid \forall t.\, h(t)(x) \in Q\}$$

- We obtain a homotopy between Cartesian liftings

$$\begin{tikzcd}
 & f^*(Y,Q) \arrow[dr, "\overline{f}\,(Y,Q)"] \arrow[dd, Rightarrow, "\overline{h}\,(Y,Q)"] & \\
h^*(Y,Q) \arrow[ur, hook] \arrow[dr, hook'] & & (Y,Q) \\
 & g^*(Y,Q) \arrow[ur, "\overline{g}\,(Y,Q)"'] &
\end{tikzcd}$$

Homotopy-Invariance for Coinductive Predicates of Hybrid Systems III/III

Example (A simple logic for hybrid systems)

- Family of modalities $\{\Box_t\}_{t \in \mathbb{R}_{\geq 0}}$
- Idea: $\Box_t \varphi$ holds at x if φ holds along the trajectory that leaves x up to (and including) time t
- Define liftings $\{U_t \colon \mathbf{cPred} \to \mathbf{cPred}\}_{t \in \mathbb{R}_{\geq 0}}$ of H with

$$U_t(X, P) = (HX, \{(\gamma, d) \in HX \mid \gamma^{\to}[0, t] \subseteq P\}) \qquad \text{and} \qquad U_t(f) = Hf$$

- Semantics of modalities in coalgebra $c \colon X \to HX$ as $\Psi_t^c = c^* \circ U_t \colon \mathbf{cPred}_X \to \mathbf{cPred}_X$
- A homotopy $h \colon Hf \circ c \Rightarrow d \circ f$ (f a homotopical coalgebra morphism) induces homotopy $\sigma \colon \Psi_t^c\left(\overline{f}\,(X, P)\right) \Rightarrow \overline{f}\,\left(\Psi_t^d(X, P)\right)$ for $P \subseteq X$ closed
- To make the logic homotopy-invariant, any such homotopy must be axiomatised

Direction 3

Develop coalgebraic modal logic further in higher categories

Wrapping Up

Challenges

1. Model HDA and directed spaces as coalgebras, which requires Vietories-like functor on $\mathbf{Top}$ or simplicial sets
2. Integration with homotopical/path categories/model categories
3. Axioms for homotopy-invariance coalgebraic modal logic
4. Cartesian fibrations in $(\infty, 1)$-categories for higher coalgebraic modal logic
5. Homotopy coherent nerve for topologically enriched fibrations
6. Obstruction theory for coalgebra via (co)homology
7. Reconciliation with directed homotopy[26]
8. Integration with type theory (synthetic $(\infty, 1)$-categories, possibly directed)

[26] Jérémy Dubut, Eric Goubault, and Jean Goubault-Larrecq. "The Directed Homotopy Hypothesis". In: *25th EACSL Annual Conference on Computer Science Logic (CSL 2016)*. Ed. by Jean-Marc Talbot and Laurent Regnier. Vol. 62. LIPIcs. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2016, 9:1–9:16. ISBN: 978-3-95977-022-4. DOI: 10.4230/LIPIcs.CSL.2016.9.

Thank you for your attention!

Thank you for your attention!