Graded Semantics and Graded Logics for Eilenberg-Moore Coalgebras

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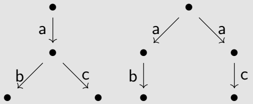
Context

Behavioural Equivalence

Often too fine grained

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Z \arrow[d] & Y \arrow[l] \arrow[d] \\
GX \arrow[r] & GZ & GY \arrow[l]
\end{tikzcd}$$



Process Semantics

Linear Time-Branching Time Spectrum

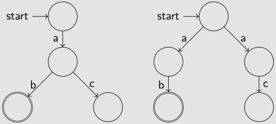
van Glabbeek '90

Graded Semantics

Coalgebraic framework for logics

Milius et al. '15

$$X \xrightarrow{\gamma} GX \xrightarrow{\alpha} M_1 X$$



Automaton Theory

Behavioural equivalence

$\neq$

Language semantics

Generalized Powerset Construction

EM-Law

Eilenberg-Moore distributive law $\zeta\colon TF \Rightarrow FT$, natural transformation compatible with the structure of T

EM-Semantics

Determinization of coalgebra $\gamma\colon X \to FTX$:

$$\gamma^{\#}\colon TX \xrightarrow{T\gamma} TFTX \xrightarrow{\zeta TX} FTTX \xrightarrow{F\mu_X} FTX$$

Behavioural equivalence in $\gamma^{\#} \approx$ Language semantics
(nondeterministic, weighted, probabilistic automata)

Silva et al. '10

Graded Monads

Graded Monads $\mathbb{M}$

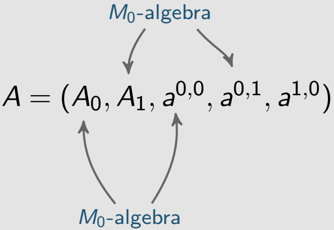
- Functors $M_n \colon \mathbf{C} \to \mathbf{C}$ for $n \in \mathbb{N}$
- Multiplications $\mu^{ij} \colon M_i M_j \Rightarrow M_{i+j}$
- Unit $\eta \colon Id \Rightarrow M_0$

$+$ monad laws (with indices)

M_n-Algebras

- Carriers A_k for $k \leq n$
- Structures $a^{ij} \colon M_i A_j \Rightarrow A_{i+j}$

$+$ algebra laws (with indices)



M_0-algebra

$$A = (A_0, A_1, a^{0,0}, a^{0,1}, a^{1,0})$$

M_0-algebra

EM-Laws Define Graded Monads

Graded Monad $\mathbb{M}_G$

- $M_n = G^n$
- All natural transformations identity

Captures (finite-depth) Behavioural equivalence.

Graded Monad $\mathbb{M}_\zeta$

- $M_n = F^n T$
- Multiplications defined by ζ
- η inherited from T

Captures *EM*-Semantics.

Graded Semantics

Definition

Graded semantics $(\mathbb{M}, \alpha\colon G \Rightarrow M_1)$

For $\gamma\colon X \to GX$ define inductively $\gamma^{(k)}\colon X \to M_k 1$:

$$\gamma^{(0)}\colon X \xrightarrow{\eta} M_0 X \xrightarrow{M_0 !} M_0 1$$

$$\gamma^{(k+1)}\colon X \xrightarrow{\alpha \cdot \gamma} M_1 X \xrightarrow{M_1 \gamma^{(k)}} M_1 M_k 1 \xrightarrow{\mu^{1k}} M_{k+1} 1$$

Graded Semantics and Terminal Chains

$$\begin{tikzcd}
1 & F1 \arrow[l, "!"'] & F^2 1 \arrow[l, "F!"'] & \cdots \arrow[l, "F^2!"'] \\
 & TX \arrow[ul, "p_0" description] \arrow[u, "p_1" description] \arrow[ur, "p_2" description] \arrow[r, "\gamma^{\#}"'] & FTX & \\
 & X \arrow[u, "\eta"] \arrow[dl, "\gamma^{(0)}" description] \arrow[d, "\gamma^{(1)}" description] \arrow[dr, "\gamma^{(2)}" description] & & \\
M_0 1 = T1 & M_1 1 = FT1 & M_2 1 = F^2 T1 & \cdots
\end{tikzcd}$$

Proposition

If $T1 \cong 1$, then graded semantics coincides with (finite-depth) EM-Semantics.

Depth-1 Graded Monads

Definition

$\mathbb{M}$ is *depth-1* if the following diagram is a coequalizer:

$$\begin{tikzcd} M_1 M_0 M_0 \arrow[r, shift left, "M_1 \mu^{00}"] \arrow[r, shift right, "\mu^{10} M_0"'] & M_1 M_0 \arrow[r, "\mu^{10}"] & M_1 \end{tikzcd}$$

Equivalent: Depth-1 algebraic theory

Lemma

If $\mathbb{M}$ depth-1

$\Rightarrow (M_n 1, M_{n+1} 1, \mu^{0,n}, \mu^{0,n+1}, \mu^{1,n})$ is free over its 0-part.

Coalgebraic Logic

Constants Modalities

Syntactic Components given by $\mathscr{L} = (\Theta, \mathscr{O}, \Lambda)$

Propositional Operators

- $\theta \in \Theta$ $\hat{\theta} \colon 1 \to \Omega$
- $p \in \mathscr{O}$ $[\![p]\!] \colon \Omega^n \to \Omega$
- $\lambda \in \Lambda$ $[\![\lambda]\!] \colon G\Omega \to \Omega$

Semantics for a coalgebra $\gamma \colon X \to GX$

$$[\![\lambda\phi]\!]_\gamma \colon X \xrightarrow{\gamma} GX \xrightarrow{G[\![\phi]\!]_\gamma} G\Omega \xrightarrow{[\![\lambda]\!]} \Omega$$

Graded Logics

Definition

$\mathscr{L}$ is a *graded logic* if

- Ω carries an M_0-algebra (Ω, o)
- $[\![p]\!] \colon \Omega^n \to \Omega$ algebra homomorphism
- $[\![\lambda]\!] = f \circ \alpha$ $(\Omega, \Omega, o, o, f)$ is an M_1-algebra

Theorem

$\mathbb{M}$ depth-1 graded monad, $\mathscr{L}$ graded logic
$\Rightarrow$ $\mathscr{L}$ invariant for $(\alpha, \mathbb{M})$

Characterizing Modalities for $\mathbb{M}_\zeta$

$$\begin{tikzcd}
FTT\Omega \arrow[r, shift left, "F\mu_\Omega"] \arrow[r, shift right, "FTo"'] & FT\Omega \arrow[rr, "\llbracket\lambda\rrbracket"] \arrow[dr, "Fo"] & & \Omega \\
& & F\Omega \arrow[ur, "h"] &
\end{tikzcd}$$

Theorem

Graded modal operators $\llbracket\lambda\rrbracket\colon FT\Omega \to \Omega$ correspond to algebra-homomorphisms $h\colon \tilde{F}(\Omega, o) \to (\Omega, o)$.

Expressivity

Depth-0 separation

$\hat{\theta}^* \colon M_0 1 \to \Omega$
jointly injective

$\llbracket \phi \rrbracket \in \Phi$

$$\begin{tikzcd}
M_1 A_0 \arrow[r, "M_1 \llbracket \phi \rrbracket"] \arrow[d, "a^{10}"] & M_1 \Omega \arrow[d, "f"] \\
A_1 \arrow[r, dashed, "\llbracket \lambda \rrbracket (\llbracket \phi \rrbracket)"] & \Omega
\end{tikzcd}$$

$f \circ \alpha \in \Lambda$

Depth-1 separation

$\forall$ A canonical (free over $(-)_0$)
$\forall$ $\Phi \subseteq A_0 \to \Omega$ jointly injective, closed under $\mathscr{O}$

$\Rightarrow \llbracket \lambda \rrbracket (\llbracket \phi \rrbracket)$ jointly injective

Theorem

If $\mathscr{L}$ is depth-0 and depth-1 separating, then $\mathscr{L}$ is expressive.

Reducing from Linear Time to Branching Time

$\mathscr{L} = (\Theta, \mathscr{O}, \Lambda)$ graded logic for $(\mathrm{id}, \mathbb{M}_\zeta)$ on FT-coalgebras

$$\Downarrow$$

$$\Lambda' = \{h \colon F\Omega \to \Omega \mid h \circ Fo \in \Lambda\}$$

$\mathscr{L}' = (\Theta, \mathscr{O}, \Lambda')$ graded logic for $(\mathrm{id}, \mathbb{M}_F)$ on F-coalgebras

Proposition

If $\mathscr{L}'$ is depth-1 separating, then $\mathscr{L}$ is depth-1 separating,

Example: Machine Functor

T arbitrary, $F = 2 \times (-)^{\Sigma}$, $\Omega = 2$

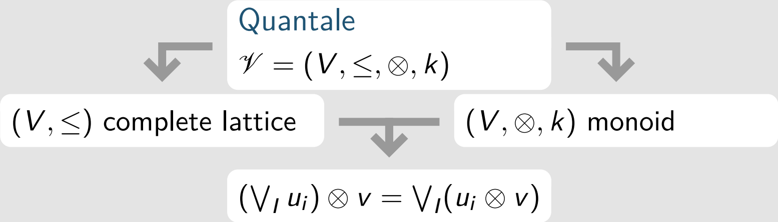
Λ': $\langle\top\rangle$ first projection, $\langle\sigma\rangle$ second projection (for $\sigma \in \Sigma$)

- Morphisms in **EM**(T) ✓

  $\Rightarrow$ Modal operators in Λ are invariant
- Depth-1 separating for $\mathbb{M}_F$ ✓

  $\Rightarrow$ $(\emptyset, \emptyset, \Lambda)$ is depth-1 separating

$\mathscr{V}$-enriched Categories



Quantale

$\mathscr{V} = (V, \leq, \otimes, k)$

$(V, \leq)$ complete lattice

$(V, \otimes, k)$ monoid

$(\bigvee_I u_i) \otimes v = \bigvee_I (u_i \otimes v)$

Example: **Met**

$M_n 1$ metric spaces,
$[\![\phi]\!] : X \to [0, 1]$

Instances

- (hemi-/pseudo-)metric spaces
- preorders (posets/setoids/sets)

Conclusion

What We Showed

- Graded semantics can capture EM-Semantics
- Invariance and expressivity follow from general results
- Quantalic generality

Future Work

- Fixpoints $\Rightarrow$ relation to regular expressions
- Quantitative Kleisli semantics