Codensity Liftings and their Applications

Shin-ya Katsumata

Kyoto Sangyo Univeristy, Japan

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Extending Set-constructions to Spatial Structures

Lifting Problem

Lift a construction on sets to sets with logical / spatial structures, e.g.

- sets with predicates,
- sets with binary relations,
- preordered sets,
- topological spaces,
- measurable spaces,
- metric spaces, etc...

Categorically, find $\dot{F}$ such that (p:forgetful)

$$\begin{array}{ccc} Sp & \xrightarrow{\dot{F}} & Sp \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{Set} & \xrightarrow{F} & \mathbf{Set} \end{array}$$

such that $F \circ p = p \circ \dot{F}$

Liftings in Semantics

Liftings have been playing two major roles in semantics:

1. Logical predicates/relations of type theories
   - [Ma&Reynolds '92,Mitchell&Scedrov '93] Lifting CC structures
   - [Hermida '93] Lifting adjunctions
   - [Hasegawa '99] Lifting SMC structures and !-modality
   - [Filinski '96, Larrecq+ '05, K. '05] Lifting monads

   Liftings are used to interpret type constructors

$$P(\tau \times \tau') \triangleq P(\tau) \dot{\times} P(\tau'), \quad P(\tau \times \tau') \triangleq P(\tau) \dot{\Rightarrow} P(\tau')$$

2. Bisimulations and modal logic
   - [Thijs '96, Hughes&Jacobs '04] Relational extension
   - [Hermida&Jacobs '98] Bisimulations through liftings
   - [Pattinson '04, Schröder '08] Modal logic and predicate liftings
   - [Baldan+ '14] Kantorovich lifting
   - [Balan+ '15,Goncharov+ '23] Lifting to $\mathbb{V}$**CAT**
   - [K.+ '15, Sprunger+ '18] Codensity lifting
   - [Bonchi+ '18] Wasserstein lifting

Bisimulations through Liftings [Hermida&Jacobs'98]

Coalgebraic Bisimulation

Let $\dot{F}$ be a lifting of F (left). A $\dot{F}$-bisimulation on $\delta : Q \to FQ$ is a $\dot{F}$-coalgebra $d : S \to \dot{F}S$ such that $pd = \delta$ (right).

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\dot{F}} & \mathbb{E} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{B} & \xrightarrow[F]{} & \mathbb{B} \end{array} \qquad\qquad \begin{array}{ccc} S & \xrightarrow{d} & \dot{F}S \\ & & \\ Q & \xrightarrow[\delta]{} & FQ \end{array}$$

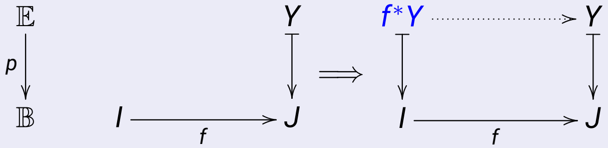
The standard definition of bisimulation relation looks like

$$\begin{aligned} \forall x, y \,.\, (x, y) \in S \Longrightarrow & (\forall x' \in \delta(x) \,.\, \exists y' \in \delta(y) \,.\, (x', y') \in S) \wedge \\ & (\forall y' \in \delta(y) \,.\, \exists x' \in \delta(x) \,.\, (x', y') \in S) \end{aligned}$$

How does this post-fixpoint style definition arise from liftings?

Fibration

... is a functor $p : \mathbb{E} \to \mathbb{B}$ admitting the inverse image operation of $\mathbb{E}$-objects along $\mathbb{B}$-morphisms:



$p : \mathbb{E} \to$ **Set**	$\mathbb{E}$-object is a set with	$\mathbb{E}$-morphism is a
EqRel $\to$ **Set**	an equiv. relation	function preserving relation
	$f^*(J, Y) = (I, \{(i, i') \mid (f(i), f(i')) \in X\})$	
EPMet $\to$ **Set**	an ext. pseudometric	nonexpansive function
	$f^*(J, d) = (I, \lambda(i, i') \,.\, d(f(i), f(i')))$	
Top $\to$ **Set**	a topology	continuous function
	$f^*(J, O) = (I, \{f^{-1}(U) \mid U \in O\})$	

$\mathbf{CLat}_\wedge$-Fibration ($\fallingdotseq$ Topological Functors)

Fiber Category

For a fibration $p : \mathbb{E} \to \mathbb{B}$ and $I \in \mathbb{B}$, define the fiber category $\mathbb{E}_I$ by

- $\mathrm{Obj}(\mathbb{E}_I) = \{X \in \mathbb{E} \mid pX = I\}$
- $\mathbb{E}_I(X, Y) = \{f \in \mathbb{E}(X, Y) \mid pf = \mathrm{id}_I\}$

The inverse image operation along $f : I \to J$ extends to $f^* : \mathbb{E}_J \to \mathbb{E}_I$.

$\mathbf{CLat}_\wedge$-Fibration

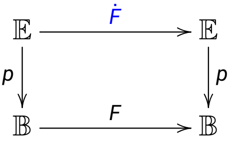
... is a fibration where each $\mathbb{E}_I$ is a complete lattice, and f^* preserves $\wedge$.
Every $\mathbf{CLat}_\wedge$-fibration is faithful (convention: $\mathbb{E}(X, Y) \subseteq \mathbb{B}(pX, pY)$).

$\mathbf{CLat}_\wedge$-fibration	Fiber over I consists of	$X \le Y$ holds if Y ... X
EqRel $\to$ **Set**	eq. relations on I	includes
EPMet $\to$ **Set**	ext. pseudometrics on I	returns smaller values than
Top $\to$ **Set**	topologies on I	is coarser than

Coalgebraic Bisimulations in Fibrations

Consider

1. a functor $F : \mathbb{B} \to \mathbb{B}$
2. a **CLat**$_\wedge$-fibration $p : \mathbb{E} \to \mathbb{B}$
3. a lifting $\dot{F} : \mathbb{E} \to \mathbb{E}$ of F

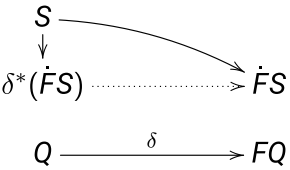


Proposition [Proposition 4.2, Hasuo+ '13]

There is a bijective correspondence between

1. $\dot{F}$-bisimulations on $\delta : Q \to FQ$
2. postfixpoints of $\delta^* \circ \dot{F} : \mathbb{E}_Q \to \mathbb{E}_Q$

The GFP $\nu(\delta^* \circ \dot{F})$ is called coinductive predicate.



Coalgebraic Bisimulations in Fibrations

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$S \subseteq \delta^*(\dot{F}S)$ expands to the standard bisimulation:

$$\begin{aligned} \forall (x, y) \in S \,.\, & (\forall x' \in \delta(x) \,.\, \exists y' \in \delta(y) \,.\, (x', y') \in S) \wedge \\ & (\forall y' \in \delta(y) \,.\, \exists x' \in \delta(x) \,.\, (x', y') \in S) \end{aligned}$$

Part I

Codensity Lifting

Extending Set-constructions to Spatial Structures

Lifting Problem

Lift a construction on sets to sets with logical / spatial structures, e.g.

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- metric spaces, etc...

Codensity liftings [K.&Sato '15][K.&Sato&Uustalu '18]

... are a method to give such liftings along fibrations. They take parameters and by varying them we obtain various liftings.

Lindley and Stark's Leapfrog Method [LS'05]

- V: set of values, C: set of computations, $[-] : V \to C$
- K: set of continuations
- $K@C$:application of a continuation to a computation

Leapfrog Method

1. Collect all continuations that send reducible terms to SN terms:

$$Red(\tau)^{\top} = \{K \mid \forall V \in Red(\tau) \,.\, K@[V] \in SN\}$$

2. Collect all computations that yield SN terms when connected with good continuations:

$$Red(T\tau) = \{C \mid \forall K \in Red(\tau)^{\top} \,.\, K@C \in SN\}$$

This refines closure operators defined similarly:

- Biorthogonality techniques in linear logic [Girard '87] and classical realizability [Krivine '09]
- $\top\top$-closure operators [Pitts '00, Abadi '00]

Semantic $\top\top$-Lifting [K.'05]

... is a method to lift a monad $(T, \eta, (-)^{\#})$: **Set** $\rightarrow$ **Set**

Parameter of the Lifting

A return type Ω and good return computations $\dot{\Omega} \subseteq T\Omega$.

Semantic $\top\top$-Lifting

1. Given a set I of values and good values $P \subseteq I$,
2. Collect all continuations sending good values to good results

$$P^{\top} = \{K : I \rightarrow T\Omega \mid \forall V \in P \,.\, K(V) \in \dot{\Omega}\}$$

3. Collect all computations that yield good results when connected with good continuations

$$P^{\top\top} = \{C : TI \mid \forall K \in P^{\top} \,.\, K^{\#}(C) \in \dot{\Omega}\}$$

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$$\begin{array}{ccc} P^{\top\top} & \longrightarrow & (P \mathrel{\dot{\Rightarrow}} \dot{\Omega}) \mathrel{\dot{\Rightarrow}} \dot{\Omega} \\ \big\downarrow \lrcorner & & \big\downarrow \\ TI & \xrightarrow{\ bind\ } & (I \Rightarrow T\Omega) \Rightarrow T\Omega \end{array} \qquad \begin{array}{c} \textbf{Pred} \\ \big\downarrow \\ \textbf{Set} \end{array}$$

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3. Collect all computations that yield good results when connected with good continuations

$\top\top$-liftings can be done in any closed-structure preserving fibrations

Can we define $\top\top$-liftings without closed structures?

Codensity Lifting [Monad: K.&Sato '15, Endofunctor: Sprunger+ '18]

Fix a $\mathbf{CLat}_\wedge$-fibration $p : \mathbb{E} \to \mathbb{B}$, functor $F : \mathbb{B} \to \mathbb{B}$ to be lifted and assume that $\mathbb{E}$ has and p preserves powers.

Parameter of the Lifting

consists of 1) $\Omega \in \mathbb{B}$, 2) $\tau : F\Omega \to \Omega$, and 3) $\dot{\Omega} \in \mathbb{E}_\Omega$.

$$\left(\begin{array}{ccc} & & \dot{\Omega} \\ & & \\ F\Omega & \xrightarrow[\tau]{} & \Omega \end{array} \right)$$

The F-algebra determines $b(\tau) : F \circ p \to Ran_{\dot{\Omega}}\Omega$.

Codensity Lifting

$$\begin{array}{ccc} [F]^{\Omega,\tau,\dot{\Omega}}X & \dashrightarrow & Ran_{\dot{\Omega}}\dot{\Omega}(X) \\ & & \\ F \circ p(X) & \xrightarrow[b(\tau)]{} & Ran_{\dot{\Omega}}\Omega(X) \end{array} \qquad \begin{array}{c} \mathbb{E} \\ \downarrow p \\ \mathbb{B} \end{array}$$

Codensity Lifting of Endofunctors [Sprunger+'18, K+'18]

Codensity Lifting in Kantorovich Style

$$[F]^{\Omega,\tau,\dot{\Omega}}X = \bigwedge_{t \in \mathbb{E}(X,\dot{\Omega})} (\tau \circ Ft)^* \dot{\Omega} \qquad (\text{using } \mathbb{E}(X,\dot{\Omega}) \subseteq \mathbb{B}(pX,\Omega))$$

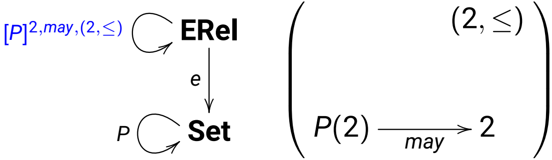
Theorem

Codensity lifting is the largest lifting that lifts τ to $\mathbb{E}$:

$$[F]^{\Omega,\tau,\dot{\Omega}} = \max\{\dot{F} \in \mathit{Lift}(F) \mid \tau \in \mathbb{E}(\dot{F}\dot{\Omega}, \dot{\Omega})\}$$

Codensity Lifting along $e : \mathbf{ERel} \to \mathbf{Set}$

We lift the powerset functor P with:



where

$$(2, \leq) = \{\bot \leq \top\}, \quad may(U) = \top \iff \top \in U$$

Proposition

The codensity lifting satisfies

$$[P]^{2,may,(2,\leq)}(Q, S) = (P(Q), \{(U, V) \mid \forall u \in U \,.\, \exists v \in V \,.\, (u, v) \in S\})$$

Examples

Table 7 Codensity lifting of functors

	Fibration $\mathbb{E} \xrightarrow{p} \mathbb{C}$	Functor $F : \mathbb{C} \to \mathbb{C}$	obs. dom. $\boldsymbol{\Omega}$	Modality τ	Lifting $F^{\boldsymbol{\Omega},\tau}$ of F
1	$\mathbf{Pre} \to \mathbf{Set}$	Powerset $\mathcal{P}$	$(2, \leq)$	$\Diamond : \mathcal{P}2 \to 2$	Lower preorder [14]
2	$\mathbf{Pre} \to \mathbf{Set}$	Powerset $\mathcal{P}$	$(2, \geq)$	$\Diamond : \mathcal{P}2 \to 2$	Upper preorder [14]
3	$\mathbf{ERel} \to \mathbf{Set}$	Powerset $\mathcal{P}$	$(2, \mathrm{Eq}_2)$	$\Diamond : \mathcal{P}2 \to 2$	(See Ex. 3.4 & 7.4)
4	$\mathbf{EqRel} \to \mathbf{Set}$	Powerset $\mathcal{P}$	$(2, \mathrm{Eq}_2)$	$\Diamond : \mathcal{P}2 \to 2$	(See Ex. 3.3 & 7.4)
5	$\mathbf{PMet}_1 \to \mathbf{Set}$	Subdistrib. $\mathcal{D}_{\leq 1}$	$([0,1], d_{[0,1]})$	$e : \mathcal{D}_{\leq 1}[0,1] \to [0,1]$	Kantorovich metric [14]
6	$\mathbf{PMet}_1 \to \mathbf{Set}$	Powerset $\mathcal{P}$	$([0,1], d_{[0,1]})$	$\inf : \mathcal{P}[0,1] \to [0,1]$	Hausdorff metric (Appx. C)
7	$U^*(\mathbf{PMet}_1) \to \mathbf{Meas}$	Sub-Giry $\mathcal{G}_{\leq 1}$	$([0,1], d_{[0,1]})$	$e : \mathcal{G}_{\leq 1}[0,1] \to [0,1]$	Kantorovich metric [14]
$8^\dagger$	$\mathbf{Pre} \to \mathbf{Set}$	Powerset $\mathcal{P}$	$(2, \leq), (2, \geq)$	$\Diamond : \mathcal{P}2 \to 2$	Convex preorder [14]
$9^\dagger$	$\mathbf{EqRel} \to \mathbf{Set}$	Subdistrib. $\mathcal{D}_{\leq 1}$	$(2, \mathrm{Eq}_2)$	$(\tau_r : \mathcal{D}_{\leq 1}2 \to 2)_{r \in [0,1]}$	(For prob. bisim., see Ex. 8.15)
$10^\dagger$	$\mathbf{Top} \to \mathbf{Set}$	$2 \times (\_)^\Sigma$	*Sierpinski* sp.	(See Ex. 6.12)	(For bisim. top., see Ex. 6.12)
$11^\dagger$	$\mathbf{BRel} \to \mathbf{Set}^2$	Any functor	$((1,1), R_2)$	Any family	(For Λ-bisim., see Sect. 8.2)
$12^\dagger$	$\mathbf{ESemi}_\mathbb{R} \to \mathbf{Vect}_\mathbb{R}$	$(\bigoplus_{a \in \Sigma}(\_)) \oplus \mathbb{R}$	$(\mathbb{R}, \lvert \cdot \rvert)$	(See Sect. 8.6)	(For bisim. seminorm, see Sect. 8.6)

The fibration $U^*(\mathbf{PMet}_1) \to \mathbf{Meas}$ is introduced in Sect. 8.5. $d_{[0,1]}$ denotes the Euclidean metric on the unit interval [0, 1]. The modality $\Diamond$ is introduced in Definition 2.12. The functions $e : \mathcal{D}_{\leq 1}[0,1] \to [0,1]$ and $e : \mathcal{G}_{\leq 1}[0,1] \to [0,1]$ both return expected values. The lower, upper and convex preorders are known for powerdomains; see e.g., [36]. The function $\tau_r : \mathcal{D}_{\leq 1}2 \to 2$ is introduced in Example 8.15. The examples marked with † involve multiple modalities and observation domains (Sect. 6)

New Generation Computing (2022) 40:403–465

Codensity Lifting along m : **EPMet** $\to$ **Set**

We lift the probability distribution functor D with:

$$[D]^{[0,\infty],E,\dot{\Omega}} \circlearrowright \mathbf{EPMet} \xrightarrow{m} D \circlearrowright \mathbf{Set} \qquad \left(\begin{array}{ccc} & & \dot{\Omega} \\ & & \\ D[0,\infty] & \xrightarrow[E]{} & [0,\infty] \end{array} \right)$$

where

$$E(\mu) = E_{x\sim\mu}[x], \quad \dot{\Omega} = ([0,\infty], \lambda(x,y) \,.\, |x-y|)$$

The codensity lifting yields the Kantorovich metric / Kantorovich lifting [Baldan+'14]

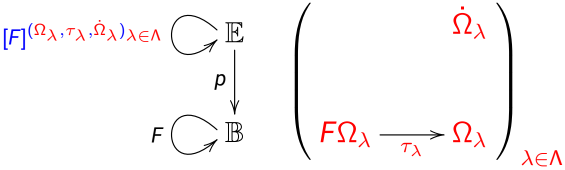
$$[D]^{[0,\infty],E,\dot{\Omega}}(Q,d) = (DQ, d^K)$$

where

$$d^K(\mu_1,\mu_2) = \sup_{t\in\mathbf{EPMet}((Q,d),\dot{\Omega})} |E_{x\sim\mu_1}[t(x)] - E_{x\sim\mu_2}[t(x)]|$$

Multiple Parameter Codensity Lifting

Consider a family of lifting parameters



Multiple Parameter Codensity Lifting

$$[F]^{(\Omega_\lambda, \tau_\lambda, \dot{\Omega}_\lambda)_{\lambda \in \Lambda}} X = \bigwedge_{\lambda \in \Lambda} [F]^{\Omega_\lambda, \tau_\lambda, \dot{\Omega}_\lambda} X$$

Λ-Bisimulations [Bakhtiari+ '17] as Codensity Lifting

Consider $F : \mathbf{Set} \to \mathbf{Set}$ and a family $\tau_\lambda : F2 \to 2$ of F-algebras.

Definition

A Λ-bisimulation between $c : Q \to FQ$ and $d : R \to FR$ is $S \subseteq Q \times R$ such that for any λ and S-coherent pair $(U : FQ \to 2, V : FR \to 2)$,

$$\tau_\lambda \circ FU \circ c = \tau_\lambda \circ FV \circ d.$$

Choose the parameter:

$$[F^2] \circlearrowright \mathbf{BRel} \xrightarrow{\;p\;} \mathbf{Set}^2 \circlearrowleft F^2 \qquad \left(\begin{array}{ccc} & & (2, 2, Eq_2) \\ & & \\ (F2, F2) & \xrightarrow[(\tau_\lambda, \tau_\lambda)]{} & (2, 2) \end{array} \right)_{\lambda \in \Lambda}$$

Theorem

S is a Λ-bisimulation over (c, d) if and only if S is a $[F^2]$-bisimulation.

Part II

Codensity Games for Bisimilarities

Komorida et al.
Codensity Games for Bisimilarity.
New Gener. Comput. 40(2): 403-465 (2022)

Outline of this Work

Classic result of characterizing the bisimilarity relation by a game

(s,t) are bisimilar in a Kripke frame $Q \to PQ$	⟷	(s,t) is a winning position in the interactive game

Can we have a similar characterization when replacing ...

1. Kripke frames with Markov chains and other transition systems?
2. bisimulation relations with simulation relations and its variants?
3. bisimulation relations with bisimulation metrics?

Contribution

We address these generalizations in fibred category theory and show

R is below the codensity bisimilarity	⟷	R is a winning position in the codensity game

Interactive Games for Bisimulation Relations

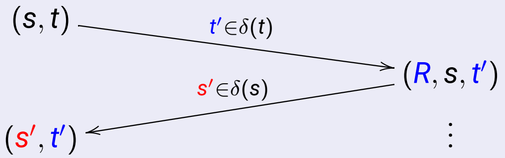
Definition

A bisimulation relation on a Kripke frame $\delta : Q \to P(Q)$ is $S \subseteq Q^2$ s.t.

$$\forall (s, s') \in S \,.\, (\forall u \in \delta(s) \,.\, \exists v \in \delta(s') \,.\, (u, v) \in S) \wedge (\forall v \in \delta(s') \,.\, \exists u \in \delta(s) \,.\, (u, v) \in S)$$

Interactive Game [Stirling '99]

Spoiler $\in Q^2$ Duplicator $\in \{L, R\} \times Q^2$



Theorem: $(s, t) \in Q^2$ is a D-winning position $\iff$ s, t are bisimilar

Generalizing the Interactive Game

The design of the interactive game depends on its logical definition (c.f. Ehrehfeucht-Fraïsse game).

Can we have a similar characterization when replacing ...

1. Kripke frames to Markov chains and other transition systems?
2. bisimulation relations to simulation relations and its variants?
3. bisimulation relations to bisimulation metrics?

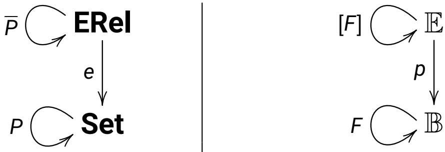
Generalizing bisimilarity games:

- General coalgebras and relation liftings [Baltag ’00, Kupke ’07]
- Markov chains and the bisimilarity relation [Desharnais+ ’08]
- Markov chains and the bisimilarity metric [Fijalkow+ ’17, König+’18]
- ... (and many more) ...

Codensity Bisimulations and Codensity Games

Can we have a similar characterization when replacing ...

1. Kripke frames to Markov chains and other transition systems?
2. bisimulation relations to simulation relations and its variants?
3. bisimulation relations to bisimulation metrics?



Kripke frame	F-coalgebra
The bisimulation relation lifting of P	Codensity lifting
Endorelations	Objects in the total category of $\mathbf{CLat}_\wedge$-fibration
Interactive game	Codensity game

Codensity Games, Conceptually

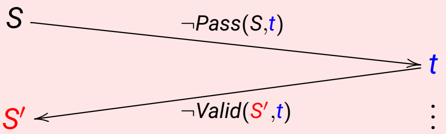
$S \subseteq Q^2$ is a codensity bisimulation relation if and only if

$$\forall t \in Test(Q) \, . \, Valid(S, t) \Longrightarrow Pass(S, t)$$

for suitable $Test, Valid, Pass$.

Codensity Game, Conceptually

Spoiler $\in P((Q)^2)$ Duplicator $\in Test(Q)$



Duplicator wins if and only if

- Spoiler gets stuck, or
- the interaction continues forever.

Codensity Bisimulations for Coalgebras

- $F : \mathbb{B} \to \mathbb{B}$, $p : \mathbb{E} \to \mathbb{B}$: $\mathbf{CLat}_\wedge$-fibration
- $\Omega \in \mathbb{C}$, $\tau : F\Omega \to \Omega$, $\dot{\Omega} \in \mathbb{E}_\Omega$: a lifting parameter

Codensity Bisimulation

An $(\Omega, \tau, \dot{\Omega})$-codensity bisimulation on $\delta : Q \to FQ$ is $S \in \mathbb{E}_Q$ s.t.

$$S \leq \delta^*([F]^{\Omega,\tau,\dot{\Omega}} S)$$

Theorem

S is an $(\Omega, \tau, \dot{\Omega})$-codensity bisimulation on $\delta : Q \to FQ$ if and only if

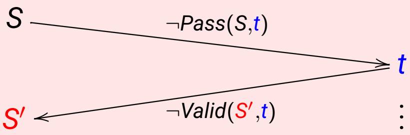
$$\forall t \in \mathbb{B}(Q, \Omega) \,.\, t \in \mathbb{E}(S, \dot{\Omega}) \Longrightarrow (\; Q \xrightarrow{\delta} FQ \xrightarrow{Ft} F\Omega \xrightarrow{\tau} \Omega \;) \in \mathbb{E}(S, \dot{\Omega})$$

Codensity Games

- $F : \mathbb{B} \to \mathbb{B}$, $p : \mathbb{E} \to \mathbb{B}$: $\mathbf{CLat}_\wedge$-fibration
- $\Omega \in \mathbb{B}$, $\tau : F\Omega \to \Omega$, $\dot{\Omega} \in \mathbb{E}_\Omega$: a lifting parameter

Codensity Game for $\delta : Q \to FQ$ with Parameter $(\Omega, \tau, \dot{\Omega})$

Spoiler $\in \mathbb{E}_Q$ Duplicator $\in \mathit{Test}(Q)$



where

$$\mathit{Test}(Q) = \mathbb{B}(Q, \Omega)$$

$$\mathit{Valid}(S, t) \iff t \in \mathbb{E}(S, \dot{\Omega}),$$

$$\mathit{Pass}(S, t) \iff \tau \circ Ft \circ \delta \in \mathbb{E}(S, \dot{\Omega})$$

Codensity Games

- $F : \mathbb{B} \to \mathbb{B}$, $p : \mathbb{E} \to \mathbb{B}$: $\mathbf{CLat}_\wedge$-fibration
- $\Omega \in \mathbb{B}$, $\tau : F\Omega \to \Omega$, $\dot{\Omega} \in \mathbb{E}_\Omega$: a lifting parameter

Theorem

The following are equivalent:

1. $W \subseteq \mathbb{E}_Q$ *is a set of Duplicator-winning positions.*
2. $\bigvee W$ *is a* $(\Omega, \tau, \dot{\Omega})$*-codensity bisimulation.*

Corollary (Characterization of the Bisimilarity Relation)

$R \in \mathbb{E}_Q$ *is D-winning* $\iff$ R *is below the* $(\Omega, \tau, \dot{\Omega})$*-codensity bisimilarity.*

Part III

Distributive Laws for Codensity-Lifted Functors

Joint work with
Mayuko Kori, Kazuki Watanabe, Jurriaan Rot

Outline of this Work

The product of deterministic automata $(Q_i, \delta_{a,i}, F_i)$:

$$(Q_1 \times Q_2, \delta_{a,1} \times \delta_{a,2}, F_1 \cap F_2)$$

Product of Bisimulations

Given bisimulation relations S_i for each automaton, can we construct a bisimulation relation on the product automaton?

Coalgebraic Reformulation

How can we extend a construction (e.g. the product) on F-coalgebras to $\dot{F}$-bisimulations?

Constructions on Coalgebras

A structure behind the product construction is a distributive law

$$\lambda : T \circ F^2 \to F \circ T,$$

yielding a functor $T_\lambda : \mathbf{Coalg}(F)^2 \to \mathbf{Coalg}(F)$.

How can we build

1. a lifting $\dot{T}$ of $T : \mathbb{B}^2 \to \mathbb{B}$ and
2. a lifting $\dot{\lambda} : \dot{T} \circ \dot{F}^2 \to \dot{F} \circ \dot{T}$ of the distributive law?

$$\begin{array}{ccc}
\mathbf{Coalg}(\dot{F})^2 & \xrightarrow{\dot{T}_{\dot{\lambda}}} & \mathbf{Coalg}(\dot{F}) \\
\downarrow & & \downarrow \\
\mathbf{Coalg}(F)^2 & \xrightarrow[T_\lambda]{} & \mathbf{Coalg}(F)
\end{array}$$

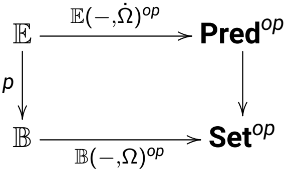
Lifting Distributive Laws

Contributions

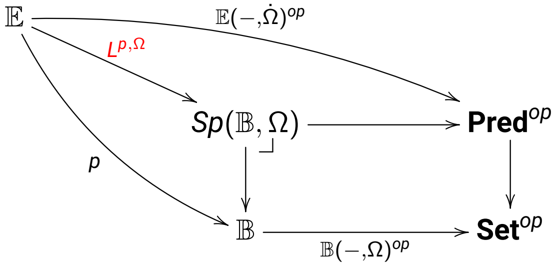
1. To lift $T : \mathbb{B}^N \to \mathbb{B}$, we generalize the codensity lifting using Beohar et al.'s decomposition of codensity liftings [Beohar+ '24].
2. We give sufficient conditions for lifting distributive laws to codensity liftings.
3. We give a composition of winning positions of codensity games using sufficient conditions (omitted)

Beohar et al.' Decomposition, Fibrationally

Given a $\mathbf{CLat}_\wedge$-fibration $p : \mathbb{E} \to \mathbb{B}$ and $\Omega \in \mathbb{B}$ and $\dot{\Omega} \in \mathbb{E}_\Omega$, we have



We name the mediating functor arising from the change-of-base $L^{p,\dot{\Omega}}$:



Remark: $L^{p,\dot{\Omega}}$ is fibred $\iff$ $\dot{\Omega}$ is c-injective in the sense of Komorida

Beohar et al.' Decomposition, Fibrationally

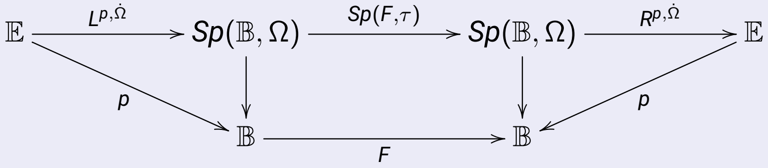
Theorem (Beohar+ '24)

1. *$L^{p,\dot{\Omega}} : \mathbb{E} \to Sp(\mathbb{B}, \Omega)$ has a right adjoint $R^{p,\dot{\Omega}} : Sp(\mathbb{B}, \Omega) \to \mathbb{E}$.*
2. *$F : \mathbb{B} \to \mathbb{B}$ and $\tau : F\Omega \to \Omega$ induces a fibred lifting of F:*

$$Sp(F, \tau) : Sp(\mathbb{B}, \Omega) \to Sp(\mathbb{B}, \Omega).$$

3. *The codensity lifting decomposes as*

$$[F]^{\Omega,\tau,\dot{\Omega}} \triangleq R^{p,\dot{\Omega}} \circ Sp(F, \tau) \circ L^{p,\dot{\Omega}} \quad \left(= \bigwedge_{k \in \mathbb{E}(-,\Omega)} (\tau \circ F(pk))^* \dot{\Omega} \right)$$



The Pullback Category $Sp(\mathbb{B}, \Omega)$

$$\begin{array}{ccc} Sp(\mathbb{B},\Omega) & \longrightarrow & \mathbf{Pred}^{op} \\ \big\downarrow \lrcorner & & \big\downarrow \\ \mathbb{B} & \xrightarrow[\mathbb{B}(-,\Omega)^{op}]{} & \mathbf{Set}^{op} \end{array}$$

- An object is a pair $(I \in \mathbb{B}, P \subseteq \mathbb{B}(I, \Omega))$, similar to topological space
- A morphism from (I, P) to (J, Q) is $f \in \mathbb{B}(I, J)$ such that $\forall k \in Q \,.\, k \circ f \in P$, similar to continuity

The assignment $(\mathbb{B}, \Omega) \mapsto Sp(\mathbb{B}, \Omega)$ extends to a 2-functor $Sp : 1/\!/\mathbf{CAT} \to \mathbf{Fib}$; here $1/\!/\mathbf{CAT}$ is the lax coslice 2-category.

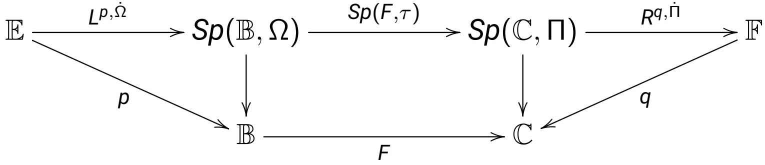
$$\begin{array}{ccc} & 1 & \\ {}^{\Omega}\swarrow & \overset{\lambda}{\Rightarrow} & \searrow^{\Pi} \\ \mathbb{C} & \xrightarrow[F]{} & \mathbb{D} \end{array}$$

Sp transfers 2-categorical structures in $1/\!/\mathbf{CAT}$ to those in **Fib**

Heterogeneous Codensity Lifting

Based on the decomposition, we define the heterogeneous codensity lifting of $F : \mathbb{B} \to \mathbb{C}$ with $\tau : F\Omega \to \Pi$ by:

$$[F, \tau] \triangleq R^{q,\dot{\Pi}} \circ Sp(F, \tau) \circ L^{p,\dot{\Omega}}.$$



When $\mathbb{B} = \mathbb{C} =$ **Set** and $\mathbb{E} = \mathbb{F} = \mathbb{V}$**Cat**, this is the same as topological lifting with $\Lambda = 1$ [Goncharov+ '23].

Instance: N-Codensity Lifting (below $N = 2$)

For $T : \mathbb{B}^2 \to \mathbb{B}$ and $p : \mathbb{E} \to \mathbb{B}$ and $\Omega \in \mathbb{B}$ and $\dot{\Omega} \in \mathbb{E}_\Omega$,

$$[T, \sigma](P_1, P_2) = \bigwedge_{k_1 \in \mathbb{E}(P_1, \dot{\Omega}), k_2 \in \mathbb{E}(P_2, \dot{\Omega})} (\sigma \circ T(pk_1, pk_2))^* \dot{\Omega}$$

Distributive Laws between Codensity Liftings

How do we obtain $\alpha : [T,\sigma] \circ [F,\tau]^N \to [F,\tau] \circ [T,\sigma]$?

$$\begin{aligned}
[T,\sigma] \circ [F,\tau]^N &= R \circ Sp(T,\sigma) \circ L \circ R \circ Sp(F^N,\tau^N) \circ L \\
&\leq R \circ Sp(T,\sigma) \circ Sp(F^N,\tau^N) \circ L \\
&\to R \circ Sp(F,\tau) \circ Sp(T,\sigma) \circ L \\
&\leq R \circ Sp(F,\tau) \circ L \circ R \circ Sp(T,\sigma) \circ L = [F,\tau] \circ [T,\sigma]
\end{aligned}$$

Theorem

The following are sufficient to lift a distributive law $\alpha : T \circ F^N \to F \circ T$.

1. α *is a distributive law in* $1//\mathbf{CAT}$:

$$\alpha : (T,\sigma) \circ (F^N,\tau^N) \to (F,\tau) \circ (T,\sigma)$$

2. *The last inequality holds (equivalently,* $Sp(T,\sigma) \circ LX$ *is* *approximating* *to* $[F,\tau]$ *for all* X *in the sense of [Komorida+ '21])*

Conclusion

$$\begin{array}{ccc} [F]^{\Omega,\tau,\dot{\Omega}}X & \dashrightarrow & Ran_{\dot{\Omega}}\dot{\Omega}(X) \\ & & \\ F \circ p(X) & \xrightarrow[b(\tau)]{} & Ran_{\dot{\Omega}}\Omega(X) \end{array} \qquad \begin{array}{c} \mathbb{E} \\ \downarrow p \\ \mathbb{B} \end{array}$$

Related Work

- Preorders on monads [K.&Sato'13]
- Relating Computational Effects by $\top\top$-Lifting [K.'13]
- Graded lifting of monads [K.'14]
- Expressivity of quantitative modal logics [Komorida+'21]

Bonchi et al. introduced Wasserstein lifting [Bonchi+ '18; see also Sprunger+ '21].

Duality in Lifting Form

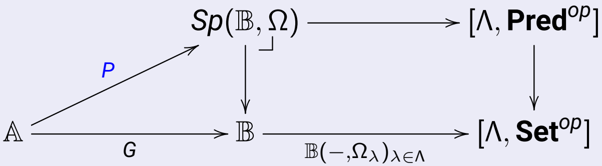
When do Wasserstein lifting and codensity lifting coincide?

Appendix: Lifting along Topological Functors

Lifting $G : \mathbb{A} \to \mathbb{B}$ along a $\mathbf{CLat}_\wedge$-fibration $p : \mathbb{E} \to \mathbb{B}$:

Parameter

... of this lifting is a family of objects $\Omega_\lambda \in \mathbb{B}, \dot{\Omega}_\lambda \in \mathbb{E}_{\Omega_\lambda}$ and



The lifting is the composite

$$\mathbb{A} \xrightarrow{P} Sp(\mathbb{B}, \Omega) \xrightarrow{R^{p,\dot{\Omega}}} \mathbb{E} .$$