Automata & coalgebras in categories of species

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What led me here

A formal theory of ~~categ~~-automata

I work in the group of **P. Sobociński** in Tallinn.

> *‘you should look into automata theory, you might like it’*

It was true.

The contact between automata theory and category theory is fertile an illustrious.

> *What can a pure category theorist give to this field? What can they learn to become better category theorists?*

A formal theory of ~~categ~~-automata

- One can organise Mealy and Moore automata in categories;
- such categories have neat characterizations and enjoy (of course) universal properties;
- a natural language to study these gadgets is 2-dimensional category theory;
- the clearest way of doing 2-category theory is formally.

But wait, *formal category theory* is what I (try to) do!

A formal theory of ~~categ~~-automata

- 'Automata' seen as categories enriched over the monoidally cocomplete poset of subsets of their input;[†]
- *realization* and *behaviour* (have univ. prop's and) define a local adjunction of bicategories (one of the weakest kind of equivalence of bicategories).

([†]worth noting: this approach is made in Italy)

UNA PROPRIETA' DEL COMPORTAMENTO
PER GLI AUTOMI COMPLETI (*)

di RENATO BETTI e STEFANO KASANGIAN (a Milano) (**)

A formal theory of ~~categ~~-automata

An insightful idea of Katis, Sabadini and Walters recognized that categories of automata organize themselves as the hom-categories of a bicategory.

Consider a monoidal category $\mathcal{K}$ as a bicategory $\Sigma\mathcal{K}$ with a single objects; take pseudofunctors $\mathbf{N} \to \Sigma\mathcal{K}$, lax natural transformations, modification.

Such bicategory can be seen as a lax analogue of a staple construction in stable homotopy theory.

A formal theory of ~~categ~~-automata

Between 1974 and 1980, R. Guitart introduces a bicategory **Mac** of (Mealy) ‘machines’ tweaking the def’n of bicategory of spans.

In a very technical paper, Guitart lays the foundation to prove that **Mac** is simply the Kleisli bicategory of the 2-monad of cocompletion under lax colimits (‘*monades des diagrammes*’).

A formal theory of ~~categ~~-automata

Recently, Bob Paré proposed the notion of a Mealy morphism as a proxy between strong functors and profunctors in any $\mathcal{V}$-enriched category $\mathcal{C}$.

The paper culminates in the impressively general and elegant result that the bicategory of $\mathcal{V}$-Mealy maps is simply the Kleisli bicategory of the lax idempotent 2-monad of $\mathcal{V}$-copower completion.

Paré generalises, in one fell swoop, KSW and Guitart's approach to every suitably nice base of enrichment.

A formal theory of ~~categ~~-automata

There is a pattern, a theme buried under these results.

Formal category theory is the best way to elucidate it.

- categories that naturally arise organizing 'machines' of sorts share a universal property of Kleisli type (they are categories of free algebras for a monad);
- the monad in question is 'of property type', i.e. it is a lax idempotent 2-monad of cocompletion under certain shapes.

Unveiling this pattern has been my interest for the last year or so.

Abstract automata

Automata

Let **C** be a strict 2-category with all finite weighted limits.

Fix a 0-cell C, an endo-1-cell $f : C \to C$ and consider as building blocks of our theory

- the inserter $u : I(f, 1_C) \to C$ or 'object of algebras' for f;
- for every $b : B \to C$ the comma object C/b (equipped with its canonical projection $C/b \to C$);
- the comma object $(f/b) \to C$.

Automata

Then, the object of (f, b)-Mealy machines is the strict 2-pullback on the left of

$$
\begin{array}{ccc}
\mathbf{Mly}(f,b) & \longrightarrow & (f/b) \\
\big\downarrow & \lrcorner & \big\downarrow \\
I(f, 1_C) & \longrightarrow & C
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbf{Mre}(f,b) & \longrightarrow & C/b \\
\big\downarrow & \lrcorner & \big\downarrow \\
I(f, 1_C) & \longrightarrow & C
\end{array}
$$

and the object of (f, b)-Moore machines is the pullback on the right.

As such, **Mly** and **Mre** are parametric functors of type

$$\mathbf{C}(C, C)^{\mathrm{op}} \times \mathbf{C}/C \longrightarrow \mathbf{C}/C$$

Automata

If $\mathbf{C} = \mathbf{Cat}$ and $b : 1 \to C$ is a single object, these definitions specialize to

- the category of Mealy automata, where objects and morphisms are of the form

$$\begin{array}{ccccc} E & \xleftarrow{d} & FE & \xrightarrow{s} & B \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle F\varphi} & & \| \\ E' & \xleftarrow[d']{} & FE' & \xrightarrow[s']{} & B \end{array}$$

- the category of Moore automata, where objects and morphisms are of the form

$$\begin{array}{ccccccc} E & \xleftarrow{d} & FE & \quad & E & \xrightarrow{s} & B \\ {\scriptstyle\varphi}\downarrow & & \downarrow & & \downarrow & & \| \\ E' & \longleftarrow & FE' & \quad & E' & \longrightarrow & B \end{array}$$

Automata

In particular, if $F_A : \mathcal{K} \to \mathcal{K}$ is the functor that tensors by an object A (an 'Alphabet'), Mealy and Moore automata are respectively diagrams of the form (E, d, s):

$$E \xleftarrow{d} A \otimes E \xrightarrow{s} B$$

and of the form

$$E \xleftarrow{d} A \otimes E, E \xrightarrow{s} B$$

$$\begin{array}{ccc} \mathbf{Mly}(A, B) & \longrightarrow & A \otimes \_/B \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{Alg}(A \otimes \_) & \longrightarrow & \mathcal{K} \end{array} \qquad \begin{array}{ccc} \mathbf{Mre}(A, B) & \longrightarrow & \mathcal{K}/B \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{Alg}(A \otimes \_) & \longrightarrow & \mathcal{K} \end{array}$$

The right level of generality is: $\mathbf{K} = \mathbf{Cat}$, ambient category is monoidal, but F is a generic endofunctor (compatible with $\otimes$).

Definition (The total categories of automata)

$(F, B) \mapsto \mathbf{Mly}(F, B)$ is a (pseudo)functor of type $\mathbf{Mly} : \mathbf{Cat}(\mathcal{K}, \mathcal{K})^{\mathrm{op}} \times \mathcal{K} \to \mathbf{Cat}$, from which we can extract a two-sided fibration

$$\mathbf{Cat}(\mathcal{K}, \mathcal{K}) \xleftarrow{p} \mathcal{M}\ell y \xrightarrow{q} \mathcal{K}$$

whose tip $\mathcal{M}\ell y$ we call the total Mealy category.

Similar considerations allow to construct the total Moore category $\mathcal{M}re$.

Automata

If $\mathcal{K}$ is monoidal its tensor functor $\_ \otimes - : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ now curries to

$$\mathcal{K} \longrightarrow \mathbf{Cat}(\mathcal{K}, \mathcal{K}) : A \mapsto A \otimes -$$

we can pullback the total Mealy fibration:

$$\begin{array}{ccc} \mathcal{M}\ell y^{\otimes} & \longrightarrow & \mathcal{M}\ell y \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{K}^{\mathrm{op}} \times \mathcal{K} & \xrightarrow[\lambda^{\mathrm{op}} \times \mathcal{K}]{} & \mathbf{Cat}(\mathcal{K}, \mathcal{K})^{\mathrm{op}} \times \mathcal{K} \end{array}$$

which gives rise to the monoidal Mealy (two-sided) fibration

$$\mathcal{K} \xleftarrow{p^{\otimes}} \mathcal{M}\ell y^{\otimes} \xrightarrow{q^{\otimes}} \mathcal{K}$$

Species

Species

Goal: focus on the category of combinatorial species.

Definition
Let S be a set and $\mathcal{V}$ a symmetric monoidal closed, complete and cocomplete, base of enrichment. The category of $(S, \mathcal{V})$-species is defined as the free symmetric monoidally cocomplete $\mathcal{V}$-category on S (regarded as discrete).

$$\underset{\text{a set}}{S} \longmapsto \underset{\text{free symmon-}\mathcal{V}\text{-cat on } S}{P\langle S\rangle} \longmapsto \underset{\text{(co)presheaves}}{\{P\langle S\rangle \to \mathcal{V}\}}.$$

In particular, the category $[P\langle 1\rangle, \mathbf{Set}]$ of $(1, \mathbf{Set})$-species is called just 'the category **Spc** of species'.

Species

More concretely, **Spc** is the category of representations of the groupoid obtained as the coproduct (in **Gpd**) $\sum_{n \geqslant 0} \mathfrak{S}_n$ of all symmetric groups.

- The species $\wp$ of subsets sends an n-set A to the 2^n-set of all its subsets;
- The species $\underline{Lin}$ of total orders sends $[n]$ to the set of total orders on $[n]$, identified with the *set* $|S_n|$ of bijections of $[n]$, over which S_n acts by left multiplication.
- The species $\underline{Sym}$ of permutations sends each finite set $[n]$ into the (carrier of the) symmetric group on n letters, S_n.
- The species $\underline{Cyc}$ of *oriented cycles* sends a finite set $[n]$ to the set of *cylic orderings* of $\{x_1, \dots, x_n\}$.

Species

Spc is fairly rich of structure:

- it is complete and cocomplete (hence it carries the Cartesian and coCartesian monoidal structures);
- it carries the pointwise monoidal product of $\mathcal{V}$;
- it carries the Day convolution monoidal structure:

$$H \circledast K := \int^{AB} HA \times KB \times \hom(A \oplus B, \_)$$

- it carries the substitution monoidal structure.

All these (closed) monoidal structures are tightly related.

Species as a differential 2-rig

Moreover, **Spc** is a (cocomplete) differential 2-rig:

Definition
A (symmetric) differential 2-rig is a (symmetric) monoidal category $(\mathcal{R}, \circledast, I)$ such that

- $X \circledast \_, \_ \circledast Y$ distribute over coproducts;
- there is an endofunctor $\partial : \mathcal{R} \to \mathcal{R}$ which is linear (preserves coproducts) and Leibniz:

$$\partial(X \circledast Y) \cong \partial X \circledast Y + X \circledast \partial Y$$

  naturally in X, Y.

It is in fact a very well-behaved differential 2-rig: the derivative functor ∂ has both a left and a right adjoint: $L \dashv \partial \dashv R$.

Automata in species as a differential 2-rig

Let $(\mathcal{R}, \otimes, I)$ be a differential 2-rig; then $\mathcal{M}\ell y_{\mathcal{R}}$ as def'd above becomes a differential 2-rig with a canonical choice of derivative functor $\bar{\partial} : \mathcal{M}\ell y_{\mathcal{R}} \to \mathcal{M}\ell y_{\mathcal{R}}$ such that $\bar{\partial}(E, d, s) = (\partial E, ..., ...)$.

Corollary

The category $\mathcal{M}\ell y_{\mathbf{Spc}}$ is a differential 2-rig such that $\bar{\partial}$ preserves all limits and colimits.

Since $\mathcal{M}\ell y_{\mathbf{Spc}}$ is also locally presentable, $\bar{\partial} : \mathcal{M}\ell y_{\mathbf{Spc}} \to \mathcal{M}\ell y_{\mathbf{Spc}}$ has a left and a right adjoint as well.

The fourfold way

Recall: $L \dashv \partial \dashv R$.

In order to study **Mly**, **Mre** based on **Spc**, one has to understand the pieces of the pullbacks before:

$$
\begin{array}{ccc}
\mathbf{Mly}_{\mathbf{Spc}}(F,B) & \longrightarrow & F\_/B \\
\downarrow & \lrcorner & \downarrow \\
\mathbf{Alg}(F) & \longrightarrow & \mathcal{K}
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{Mre}_{\mathbf{Spc}}(A,B) & \longrightarrow & \mathcal{K}/B \\
\downarrow & \lrcorner & \downarrow \\
\mathbf{Alg}(F) & \longrightarrow & \mathcal{K}
\end{array}
$$

and in particular, co/algebras for left adjoints $F(\dashv \#)$, so that co/limits in $\mathbf{Mly}(F,B)$, $\mathbf{Mre}(F,B)$ are particularly easy to compute.

There are various choices for $F(\dashv \#)$: the functor L; the functor ∂; the functor $L\partial$; the functor ∂L.

This could be called the fourfold way.

The fourfold way

The category $\mathbf{Mly}_{\mathbf{Spc}}(L, B)$ is modeled over the category $\mathbf{Spc}^L$, equivalently described as

- the category of endofunctor algebras for $L = y[1] \circledast \_$;
- the category of endofunctor coalgebras for ∂;
- the Eilenberg–Moore category of the monad $\underline{\mathit{Lin}} \circledast \_$;
- the coEilenberg–Moore category of the comonad $\{\underline{\mathit{Lin}}, -\}_{\mathrm{Day}}$.

Note the def'n of $L \dashv \partial$; $\{\underline{\mathit{Lin}}, -\}_{\mathrm{Day}}$ is the internal hom for Day convolution.

The fourfold way

One can study $L\partial, \partial L$ as mere endofunctors (plain dynamics) or qua comonad and monad respectively (monadic dynamics), taking Eilenberg–Moore algebras instead of endofunctor algebras in the definition of $\mathbf{Mly}(T, B)$.

Co/monadic dynamics seems a relatively unexplored part of the theory, and rather unrewarding for a variety of reasons. (There can be a conceptual explanation of this.)

The fourfold way

First step: explicit formulas for the monad and comonad in study.

- $L\partial H$ acts as $y[1] \circledast \partial H$; a structure of type $L\partial H$ on a finite set A chooses a point of A, and an H-structure on the complement of that point.
- $R\partial H$ acts as $A \mapsto \prod_{a\in A} H[(A \setminus \{a\}) \sqcup \{\bullet\}]$, i.e. as $A \mapsto HA^A$; a structure of type $R\partial H$ on a finite set A chooses an H-structure on A for every $a \in A$. With a similar reasoning,
- $\partial LH = \partial(y[1] \circledast H)$ is the functor $H + L\partial H$;[1] note in particular that the unit of the monad ∂L is the first coproduct injection.
- ∂RH acts as $A \mapsto H[A]^A \times H[A] = R\partial H[A] \times H[A]$; note in particular that the counit of the comonad is the second product projection.

[1]This gives rise to the evocative formula: $[\partial, L] = \partial L - L\partial = 1$, i.e. to the canonical commutation relation between position and momentum (up to a sign).

The fourfold way

Recipe:

- fix an interesting species H;
- outline what a $L\partial$-algebra structure on H amounts to;
- some times it might be easier to describe a $R\partial$-coalgebra structure;
- deduce interesting properties of the category $\mathbf{Mly}(L\partial, B)$ for different choices of output object B;
- do the same for ∂L-coalgebras (=∂R-algebras).

What now?

- **Differential equations.** The canonical commutation $[\partial, L] = \partial L - L\partial = 1$ valid in Joyal's virtual species suggests the existence of a categorified 'Heaviside distribution' Θ with the property that the colimit of F weighted by Θ is a solution of the differential equation $\partial G = F$ on species.
- **Even more abstract machines.** Defining 'machines' as limit diagrams obtained from diagrams

$$X \xrightarrow{1} X \xleftarrow{f} X \xrightarrow{f} X \xleftarrow{b} B$$

  is powerful: one can define analogues for $\mathbf{Mly}(A, B)$, $\mathbf{Mre}(A, B)$ enriched over a generic monoidal base $\mathcal{W}$, so that now there is a metric space $\mathbf{Mly}_{(X,d)}(f, b)$ associated to every nonexpansive map $f : X \to X$ and point $b \in X$.