# Coinductive control of inductive data types

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# Outline

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# Overview

## Theorem (N.-Péroux)

The category of algebras over an *accessible, lax symmetric monoidal* endofunctor on a *locally presentable, symmetric monoidal closed* category is enriched over the category of coalgebras of the same endofunctor.

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There are many examples, including polynomial endofunctors with extra structure.

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## Gain

Get more control over algebras

- Get more "initial algebras" (e.g. generalized W-types)

# Outline

# Natural numbers

## Syntax

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Inductive N : Type :=
| 0 :  N
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## Categorical semantics

1. Consider the endofunctor $X \mapsto 1 + X$ on Set.
2. An *algebra* is a set $X$ together with $\langle 0_X, s_X \rangle : 1 + X \to X$.
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## Coinductive data types and coalgebras

1. A *coalgebra* is a set $X$ together with $X \to 1 + X$.
2. The terminal coalgebra is $\mathbb{N}^\infty$.

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# Outline

# Previous work on coalgebraic enrichment

## Univeral measuring coalgebra (Wraith, Sweedler 1968)

For $k$-algebras $A$ and $B$, there is a $k$-coalgebra $\underline{\mathrm{Alg}}(A, B)$

- which underlies an enrichment of $k$-algebras in $k$-coalgebras
- whose *set-like elements*[1] are in bijection with $\mathrm{Alg}(A, B)$.

Taking $B := k$, one gets the *dual* $\underline{\mathrm{Alg}}(A, k)$ of $A$.

## Extensions

- Anel-Joyal 2013 (dg-algebras)
- Hyland-Franco-Vasilakopoulou 2017 (monoids)
- Vasilakopoulou 2019 ($\mathcal{V}$-categories)
- Péroux 2022 ($\infty$-algebras of an $\infty$-operad)
- McDermott-Rivas-Uustalu 2022 (monads)

[1] those $c \in \underline{\mathrm{Alg}}(A, B)$ s.t. $\Delta c = c \otimes c$ and $\epsilon(c) = 1_A$

# Enriched categories

## Definition

An *enrichment* of a category $\mathcal{C}$ in a monoidal category $\mathcal{V}$ consists of

- a functor $\underline{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$
- a morphism $\mathbb{I} \to \underline{\mathcal{C}}(A, A)$ for each $A \in \mathrm{ob}\ \mathcal{C}$
- a morphism $\underline{\mathcal{C}}(A, B) \otimes \underline{\mathcal{C}}(B, C) \to \underline{\mathcal{C}}(A, C)$ for $A, B, C \in \mathrm{ob}\ \mathcal{C}$
- an isomorphism $\mathcal{V}(\mathbb{I}, \underline{\mathcal{C}}(A, B)) \cong \mathcal{C}(A, B)$ for $A, B \in \mathrm{ob}\ \mathcal{C}$.

such that ...

## Remark

Monoidal *closed* means enriched in itself.

# Outline

Overview

Categorical W-types

Forerunners

**Endofunctors**

# Measuring in general

Fix a *locally presentable, symmetric monoidal closed* category $\mathcal{C}$ and an *accessible, lax symmetric monoidal*endofunctor $F$.

## Measuring

For algebras $(A, \alpha), (B, \beta)$ a *measure* $(A, \alpha) \to (B, \beta)$ is a coalgebra $(C, \chi)$ together with a morphism $\phi : C \to \underline{\mathcal{C}}(A, B)$ satisfying:

$$\begin{array}{ccccc} & & FC \xrightarrow{F(\phi)} F(\underline{\mathcal{C}}(A,B)) & \longrightarrow & \underline{\mathcal{C}}(FA, FB) \\ & \overset{\chi}{\nearrow} & & & \downarrow \beta \\ C & & & & \\ & \underset{\phi}{\searrow} & & & \\ & & \underline{\mathcal{C}}(A,B) & \xrightarrow{\quad\alpha\quad} & \underline{\mathcal{C}}(FA, B) \end{array}$$

The *universal measure* $\underline{\mathrm{Alg}}(A, B)$ is the terminal one.

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## Theorem (N.-Péroux)

The universal measure $\underline{\mathrm{Alg}}(A, B)$ always exists, and these are the hom-coalgebras of an enrichment of $\mathrm{Alg}(F)$ in $\mathrm{CoAlg}(F)$.

# Measuring for the natural numbers

## Measuring

For algebras $A, B$, a *measure* $A \to B$ is a coalgebra $C$ together with a function $C \to A \to B$ such that

- $f_c(0_A) = 0_B$ for all $c \in C$;
- $f_c(a+1) = 0_B$ for all $[\![c]\!] = 0$ and for all $a \in A$;
- $f_c(a+1) = f_{c-1}(a) + 1$ for $[\![c]\!] \geqslant 1$ and for all $a \in A$.

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The *universal measure* $\underline{\mathrm{Alg}}(A, B)$ is the terminal measure $A \to B$.

What is this?

# Set-like elements in general

## Definition

The *set-like elements* are

$$\mathbb{1} \to \underline{\mathrm{Alg}}(A, B) \qquad \text{in } \mathrm{CoAlg}(F)$$

i.e., elements of $\mathrm{Alg}(A, B)$.

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- The *points* of $\mathrm{Alg}(A, B)$ are total algebra homomorphisms $A \to B$.

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That is

- The *points* of $\mathrm{Alg}(A, B)$ are total algebra homomorphisms $A \to B$.
- If we're considering $(\mathrm{Set}, \times, *)$, the underlying set of $\mathbb{I}$ is $*$, so these are 'special' elements of the underlying set of $\mathrm{Alg}(A, B)$.

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so $f_*$ is a total algebra homomorphism

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## Example

$$\mathrm{Alg}(\mathbb{N}, A) \cong *$$

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So denote the elements of $\underline{\mathsf{Alg}}(\mathbb{N}, A)$ by

- $f_0$
- $f_1$

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- $f_\infty$

**Measuring**

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**Definition**

So we call elements of the underlying of $\underline{\mathrm{Alg}}(A, B)$ *n-partial algebra homomorphisms*.

# What are the non-set-like elements?

- Let $\mathbb{n}$ denote the quotient of $\mathbb{N}$ by $m = n$ for all $m \geqslant n$.
- Let $\mathbb{n}^\circ$ denote the subobject of $\mathbb{N}^\infty$ consisting of $\{0, \dots, n\}$.

## Example

$$\mathrm{Alg}(\mathbb{n}, A) \cong \begin{cases} * & \text{if } n_A = m_A \text{ for all } m \geqslant n; \\ \varnothing & \text{otherwise.} \end{cases}$$

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$$\underline{\mathrm{Alg}}(\mathbb{n}, A) \cong \begin{cases} \mathbb{N}^\infty & \text{if } n_A = m_A \text{ for all } m \geqslant n; \\ \mathbb{n}^\circ & \text{otherwise.} \end{cases}$$

- So there is at least always an $n$-partial homomorphism out of $n$ (which is unique).

# What can we do with this?

Generalize W-types, i.e., initial algebras.

## $C$-initial objects

For a coalgebra $C$, a *$C$-initial algebra* is an algebra $A$ such that for all other algebras $B$ there is a unique

$$C \to \underline{\mathrm{Alg}}(A, B).$$

## Initial object

An initial object in a category $\mathcal{C}$ is an object $A$ such that for all other algebras $B$ there is a unique

$$* \to \mathcal{C}(A, B).$$

# $C$-initial objects for the natural numbers

## Examples

For the natural-numbers endofunctor:

- $\mathbb{N}$ is the $\mathbb{I}$-*initial algebra*
- $\mathbb{N}$ is the $\mathbb{N}^\infty$-*initial algebra*

# $C$-initial objects for the natural numbers

## Examples

For the natural-numbers endofunctor:

- $\mathbb{N}$ is the $\mathbb{I}$-*initial algebra*
- $\mathbb{N}$ is the $\mathbb{N}^\infty$-*initial algebra*

- $\mathbb{I}$- (or $\mathbb{N}^\infty$-) initial means initial with respect to total algebra homomorphisms

## Theorem

$\mathbb{n}$ is the $\mathbb{n}^\circ$-*initial algebra*

- $\mathbb{n}^\circ$-initial means initial with respect to partial algebra homomorphisms

# Examples

(Endofunctors on a locally presentable symmetric monoidal category)

- (id) The identity endofunctor
- ($A$) The constant endofunctor at fixed commutative monoid $A$
- ($GF$) The composition of two instances
- ($F \otimes G$) The tensor of two instances ($\mathcal{C}$ closed)
- ($F + G$) The coproduct of an instance $F$ and an '$F$-module' $G$
- ($\mathrm{id}^A$) The exponential $\mathrm{id}^A$ at object $A$ ($\mathcal{C}$ cartesian closed)
- ($W$-type) The polynomial endofunctor associated to a morphism $f : X \to Y$, given a commutative monoid structure on $Y$ and an oplax symmetric monoidal structure on the preimage functor $f^{-1} : C \to \mathrm{Set}$ ($\mathcal{C} = \mathrm{Set}$)
- (d.e.s.) A discrete equational system (monoidal structure on $\mathcal{C}$ is cocartesian, $\mathcal{C}$ has binary products that preserve filtered colimits)

# Summary

We have

- that algebras are enriched in coalgebras (under certain hypotheses)
- an interpretation as notion of partial algebra homomorphism (especially in the case $N$)
- many examples
- a more refined notion of initial algebra

# Future work

- Work out more of the examples in detail
- Understand $C$-initial algebras in more examples and in general
- Understand if this extra structure is useful for programming languages
- Understand if there is a connection with domain theory

# Thank you!