Composite Theories and Distributive Laws

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VRIJE UNIVERSITEIT AMSTERDAM

This talk

Monads $\Longleftrightarrow$ Algebraic Theories

Distributive laws $\overset{\text{Cheng'20}}{\underset{\text{Piróg, Staton'17}}{\Longleftrightarrow}}$ Composite Theories

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Weak distributive laws $\overset{?}{\Longleftrightarrow}$ *Weak composite theories?*

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Distributive laws $\Longrightarrow$ Zwart'20 $\Longleftarrow$ Composite Theories

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Preliminaries

Monads

Monads are

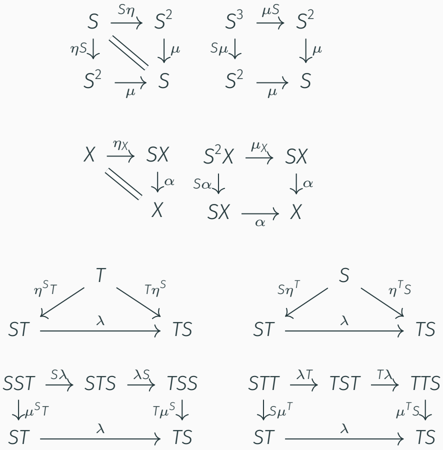
- functor $S : C \to C$
- unit $\eta : id \Rightarrow S$
- multiplication $\mu : SS \Rightarrow S$

S-algebras are

- $(X, SX \xrightarrow{\alpha} X)$

Distributive laws are

- $\lambda : ST \Rightarrow TS$



Algebraic theories

Algebraic theories $\mathbb{S}$ are

- ***signature*** $\Sigma_\mathbb{S} = \{f^{(2)}, g^{(1)}, \ldots\}$
- ***equations*** $E_\mathbb{S} = \{(s,t), \ldots\}$

$\mathbb{S}$-***algebras*** are $(X, \{X^2 \xrightarrow{[\![f]\!]} X, \ldots\})$ satisfying equations:

$$[\![s]\!]_\sigma = [\![t]\!]_\sigma \qquad \forall (s,t) \in E, \forall \text{var. assign. } \sigma$$

$$\text{Set} \underset{U}{\overset{\mathcal{T}(\Sigma_\mathbb{S},-)/E_\mathbb{S}}{\rightleftarrows}} \text{Alg}(\mathbb{S}) \quad \Longrightarrow \quad \textit{free algebra monad } T_\mathbb{S}$$

Algebraic presentation

$\mathbb{S}$ is an ***algebraic presentation*** of S if

$$T_{\mathbb{S}} \cong S \qquad \text{or equivalently} \qquad \mathbf{Alg}(\mathbb{S}) \cong_{\text{conc}} \mathbf{EM}(S)$$

For instance

$$\begin{array}{rccccl} \text{list} & [x_1, \ldots, x_n] & \Longleftrightarrow & x_1 \cdot \ldots \cdot x_n & \text{in Monoid} \\ \text{set} & \{x_1, \ldots, x_n\} & \Longleftrightarrow & x_1 \cdot \ldots \cdot x_n & \text{in JoinSemiLattices} \\ \text{distribution} & px + (1-p)y & \Longleftrightarrow & x \oplus_p y & \text{in ConvexAlgebras} \\ \vdots & & \Longleftrightarrow & & \vdots \end{array}$$

Rewriting

String rewriting	Term rewriting
$ab \to c$	$f(x, g(y)) \to f(x, x)$
$bc \to a$	$h(x) \to f(g(x), g(x))$
$ca \to a$	

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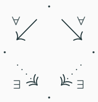
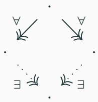
Termination (SN)	Local Confluence (WCR)	Confluence (CR)
$\cdot \to \cdot \to \ldots \to \cdot \not\to$		

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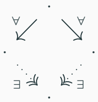
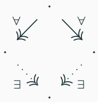
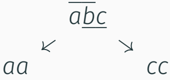
$\cdot \to \cdot \to \ldots \to \cdot \not\to$		

- SN ∧ CR $\implies$ terms rewrite to unique normal forms (no more steps).
- **Newman's Lemma:** $SN \wedge WCR \implies CR$
- *Critical pairs* are rules that overlap



- **Critical Pair's Lemma:** $WCR \iff$ all critical pairs converge.

Composite theories

Example

Example: Monoids, Abelian group, and Rings.

$$\Sigma_{\text{Mon}} := \{\cdot^{(2)}, 1^{(0)}\} \qquad \Sigma_{\text{AbGrp}} := \{0^{(0)}, +^{(2)}, -^{(1)}\}$$

$$E_{\text{Mon}} := \left\{\begin{array}{c}(x\cdot y)\cdot z = x\cdot(y\cdot z),\\ 1\cdot x = x,\\ x\cdot 1 = x\end{array}\right\} \qquad E_{\text{AbGrp}} := \left\{\begin{array}{c}(x+y)+z = x+(y+z),\\ x+(-x) = 0,\\ x+y = y+x,\\ x+0 = x\end{array}\right\}$$

Then:

$$\Sigma_{\text{Ring}} := \Sigma_{\text{Mon}} \uplus \Sigma_{\text{AbGrp}} \qquad E_{\text{Ring}} := E_{\text{Mon}} \cup E_{\text{AbGrp}} \cup \left\{\begin{array}{c}x(y+z) = (xy)+(xz),\\ (y+z)x = (yx)+(zx)\end{array}\right\}.$$

We can distribute everything, so every Ring term is equal to an AbGrp term with Monoid terms substituted.

Composite Theories of $\mathbb{T}$ after $\mathbb{S}$

Definition

Algebraic theories $\mathbb{S}, \mathbb{T} \subseteq \mathbb{U}$.

- $\mathbb{U}$-term is ***separated*** if of the form $t[s_x/x]$.
- Two separated terms $t[s_x]$ and $t'[s'_y]$ are ***equal modulo*** $(\mathbb{S}, \mathbb{T})$ if

$$\overline{t[\overline{s_x}^{\mathbb{S}}]}^{\mathbb{T}} = \overline{t'[\overline{s'_y}^{\mathbb{S}}]}^{\mathbb{T}} \quad \text{(in } TS\mathcal{V}\text{)}$$

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- $\mathbb{U}$ is a ***composite theory*** of $\mathbb{T}$ after $\mathbb{S}$ if
  - every $\mathbb{U}$-term u has a ***separation*** $u =_{\mathbb{U}} t[s_x/x]$
  - any $t[s_x] =_{\mathbb{U}} t'[s'_x] \implies t[s_x]$ and $t'[s'_x]$ must be equal modulo $(\mathbb{S}, \mathbb{T})$.

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Example of equal modulo $(\mathbb{S}, \mathbb{T})$: $\overline{0}^{\text{AbGrp}} = \overline{(\overline{1 \cdot x}^{\text{Mon}}) + (-(\overline{x \cdot 1}^{\text{Mon}}))}^{\text{AbGrp}}$.

- $x + (-x) =_{\text{AbGrp}} 0$
- $x \cdot 1 =_{\text{Monoid}} 1 \cdot x$

Dist. Laws $\iff$ Composite Th.

$\Longleftarrow$ proof

Theorem (D.L $\Longleftarrow$ Comp. Th. Zwart'20)

Monads S, T presented by theories $\mathbb{S}, \mathbb{T}$.
Given composite theory $\mathbb{U}$ of $\mathbb{T}$ after $\mathbb{S}$, then

$$\lambda_{\mathcal{V}} : ST\mathcal{V} \to TS\mathcal{V} :$$
$$\overline{s[\overline{t_x}^{\mathbb{T}}/x]}^{\mathbb{S}} \mapsto \overline{t'[\overline{s'_x}^{\mathbb{S}}/x]}^{\mathbb{T}} \text{ (a separation)}$$

is a distributive law with monad $T \circ_\lambda S$ presented by $\mathbb{U}$.

Proof.

λ well-defined by equality modulo $(\mathbb{S}, \mathbb{T})$.
Straightforward but tedious. □

$\Longrightarrow$ proof

Theorem (D.L $\Longrightarrow$ Comp. Th.)

Monads S, T presented by theories $\mathbb{S}, \mathbb{T}$.

Distributive law $\lambda : ST \Rightarrow TS$.

$$E_\lambda := \left\{ (s[t_x/x], t[s_y/y]) \mid \lambda_\mathcal{V}\left(\overline{s[\overline{t_x}^{\mathbb{T}}/x]}^{\mathbb{S}}\right) = \overline{t[\overline{s_y}^{\mathbb{S}}/y]}^{\mathbb{T}} \right\}.$$

$$\Sigma_{\mathbb{U}^\lambda} := \Sigma_\mathbb{S} \uplus \Sigma_\mathbb{T},$$

$$E_{\mathbb{U}^\lambda} := E_\mathbb{S} \cup E_\mathbb{T} \cup E_\lambda.$$

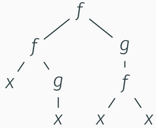
Then, $\mathbb{U}^\lambda$ is a composite theory of $\mathbb{T}$ after $\mathbb{S}$.

More tools needed for the proof.

Tools needed

- Define function type that give $\mathbb{U}^\lambda$-terms a corresponding $\{S, T\}^*\mathcal{V}$.
  E.g. for $f^{(2)} \in \Sigma_{\mathbb{S}}$, and $g^{(1)} \in \Sigma_{\mathbb{T}}$:

$u =$



$$\text{type}(u) := STS\mathcal{V}.$$

$$\overline{u}^{STS} := \overline{f(f(\overline{\overline{x}^{\mathbb{S}}}^{\mathbb{T}}, \overline{g(\overline{x}^{\mathbb{S}})}^{\mathbb{T}}), \overline{g(\overline{f(x,x)}^{\mathbb{S}})}^{\mathbb{T}})}^{\mathbb{S}}.$$

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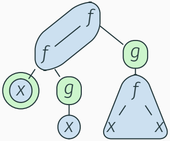

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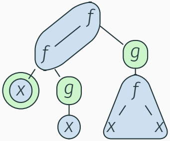

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- Apply $\lambda, \mu^{S}, \mu^{T}$ to $\overline{u}^{STS}$ until we reach $TS\mathcal{V}$, $T\mathcal{V}$ or $S\mathcal{V}$

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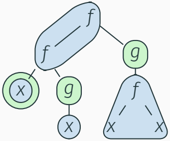
$u =$ 

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Definition (c.f. *rewrite category* Kozen'19)

Functor rewriting system (FRS) $(\Sigma, \mathcal{R})$ consist of

- $\Sigma := \{F_i \mid i \in I\}$, set of functors
- $\mathcal{R} := \{\alpha_j : w_j \to w'_j \mid w_j, w'_j \in \Sigma^*, j \in J\}$, set of natural transformations

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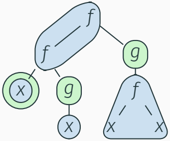

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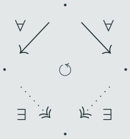
We define: $\mathcal{R}^{sep} = (\Sigma, R)$, where

- $\Sigma := \{S, T\}$
- $R := \{\lambda : ST \to TS,\ \mu^S : SS \to S,\ \mu^T : TT \to T\}$

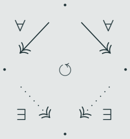
Properties of FRS

Definition

Local Confluence-commuting (WCR⟲)



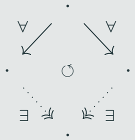
Confluence-commuting (CR ⟲)



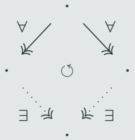
Properties of FRS

Definition

Local Confluence-commuting (WCR$\circlearrowleft$)



Confluence-commuting (CR $\circlearrowleft$)



Lemma (FRS Newman's Lemma)

$SN \wedge WCR \circlearrowleft \implies CR \circlearrowleft$

Lemma (FRS Critical Pair's Lemma)

$WCR \circlearrowleft \iff$ *all critical pairs converge with a commuting diagram.*

Properties of $\mathcal{R}^{sep}$

Lemma

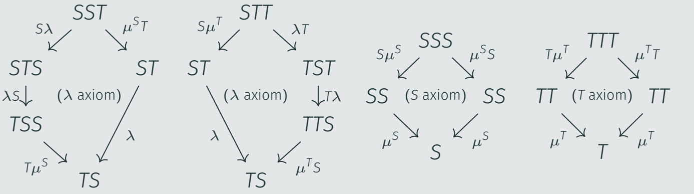
$\mathcal{R}^{sep} = (\{S, T\}, \{\lambda, \mu^S, \mu^T\})$ *is* SN *and* CR ⟳.

Proof.

- SN: polynomial interpretation over $\mathbb{N}$. $[\![S]\!](x) := 2x + 1$, $[\![T]\!](x) := x + 1$

$$[\![ST]\!](x) = 2x + 3 > 2x + 2 = [\![TS]\!](x),$$
$$[\![SS]\!](x) = 4x + 3 > 2x + 1 = [\![S]\!](x),$$
$$[\![TT]\!](x) = x + 2 > x + 1 = [\![T]\!](x).$$

- WCR⟳: exactly 4 critical pairs:

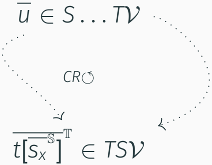


□

Consequences of $\mathcal{R}^{sep}$ being SN and CR ↻

For $\mathbb{U}^\lambda$-term u, define $\mathrm{sep}(u)$ separated and $u =_{\mathbb{U}^\lambda} \mathrm{sep}(u)$, thanks to:

- Unique normal form (TS, S or T)
- Any paths to normal form are equal.



Lemma

Every $\mathbb{U}^\lambda$-term can be separated. ✓

Finishing the proof

Lemma

Any two separated terms equal in $\mathbb{U}^\lambda$ are equal modulo $(\mathbb{S}, \mathbb{T})$.

Sketch of proof.

Induction on proof-tree.

Each $u = u'$, we prove $\text{sep}(u), \text{sep}(u')$ are equal modulo $(\mathbb{S}, \mathbb{T})$

$\dfrac{(s,t) \in E_{\mathbb{S}}}{s = t}$ Ax.	E.g. $\overline{\text{sep}(s_1)}^{\mathbb{S}} = \overline{s_1}^{\mathbb{S}} = \overline{s_2}^{\mathbb{S}} = \overline{\text{sep}(s_2)}^{\mathbb{S}}$
$\dfrac{}{u = u}$ Refl.	$\overline{\text{sep}(u)}^{TS} = \overline{\text{sep}(u)}^{TS}$
$\dfrac{u_1 = u_2}{u_2 = u_1}$ Sym.	IH = goal

Continued

Sketch of proof continued.

$$\frac{u_1 = u_2 \qquad u_2 = u_3}{u_1 = u_3}\ \text{Trans.}$$	$$\overline{\mathsf{sep}(u_1)}^{TS} = \overline{\mathsf{sep}(u_2)}^{TS} = \overline{\mathsf{sep}(u_3)}^{TS}$$
$$\frac{u_1 = u_1' \qquad \dots \qquad u_n = u_n'}{\mathsf{op}(u_1, \dots, u_n) = \mathsf{op}(u_1', \dots, u_n')}\ \text{Cong.}$$	E.g. when $\mathsf{op} \in \Sigma_{\mathbb{T}}$: $$\begin{aligned} &\overline{\mathsf{sep}(\mathsf{op}(u_1, \dots, u_n))}^{TS} \\ &= \mu^T S(\overline{\mathsf{op}(t_1[s_1], \dots, t_n[s_n])}^{TTS}) \\ &\overset{\text{IH}}{=} \mu^T S(\overline{\mathsf{op}(t_1'[s_1'], \dots, t_n'[s_n'])}^{TTS}) \\ &= \overline{\mathsf{sep}(\mathsf{op}(u_1', \dots, u_n'))}^{TS} \end{aligned}$$
$$\frac{u = u'}{u[f] = u'[f]}\ \text{Subt.}\ .$$	Separate substitution: $f(y) =_{\mathbb{U}^\lambda} t_y[s_z]$. $$\begin{aligned} &\overline{\mathsf{sep}(u[f])}^{TS} \\ &= \mu^{TS}(\overline{t[s_x[t_y[s_z]]]}^{TSTS}) \\ &\overset{\text{IH}}{=} \mu^{TS}(\overline{t'[s_x'[t_y[s_z]]]}^{TSTS}) \\ &= \overline{\mathsf{sep}(u'[f])}^{TS} \end{aligned}$$

□

Presentation of $\mathbb{U}^\lambda$

Theorem (Zwart'20)

The monad $T \circ_\lambda S$ is presented by $\mathbb{U}^\lambda$.

Proof updated.

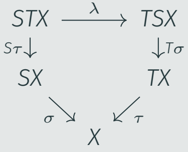
Shortcut $\mathbf{EM}(TS) \cong_{\text{conc}} \mathbf{Alg}(\lambda)$.

λ-algebras are triples (X, σ, τ), such that

- (X, σ) is an S-algebra
- (X, τ) is a T-algebra

λ-algebra morphisms are $f : X \to Y$ such that

- $f : (X, \sigma_X) \to (Y, \sigma_Y)$ is S-algebra morphism.
- $f : (X, \tau_X) \to (Y, \tau_Y)$ is T-algebra morphisms.



□

Axiomatisation $\mathbb{U}^\lambda$

Axiomatisation example

Main theorem requires E_λ to contain all distributivity equations.

Example (Ring)

$$\lambda : \mathrm{Mon} \cdot \mathrm{AbGrp} \Rightarrow \mathrm{AbGrp} \cdot \mathrm{Mon}$$

$$\begin{array}{llll} x = x & (x+y)z = xz + yz & x \cdot 0 = 0 & (-x)y = -(xy) \\ x = x + 0 & x(y+z) = xy + xz & 0 \cdot x = 0 & x(-y) = -(xy) \\ x = 0 + x & (x+y)(z+w) = xz + xw + yz + yw & 0 \cdot x = 0 + 0 & (-x)(-y) = xy \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

Goal: Find minimal axiomatisation $\Longrightarrow$ general tools

Layers

Definition

ST-layers of term $s[t_x/x] \in \Sigma_{\mathbb{S}}^*\Sigma_{\mathbb{T}}^*\mathcal{V}$, are pair $(m, n) \in \mathbb{N}^2$

$$\begin{cases} m := \text{depth}(s) \\ n := \max\{\text{depth}(t_x) \mid x \in \text{var}(s)\} \end{cases} \qquad \text{(const. depth 1)}$$

Example (Ring, $\mathbb{S} = \text{Mon}, \mathbb{T} = \text{AbGrp}$)

ST-Layers	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(0,2)$
Examples	x	0	1	$x \cdot 0$	$x + 0$
	y	$x + y$	$x \cdot y$	$(x+y)\cdot(y+z)$	$(x+y)+z$

Lemmas

Lemma

For all $E' \subseteq E_\lambda$ such that for each $f^{(n)} \in \Sigma_\mathbb{S}$, $g^{(m)} \in \Sigma_\mathbb{T}$ and each $i \in \{1, \ldots, n\}$, E' contains one equation of the form $l = r$, where

- $l = f(x_1, \ldots, x_{i-1}, g(\vec{y}), x_{i+1}, \ldots, x_n)$
- $r \in \lambda_\mathcal{V}(\bar{l}^{ST})$,

If the TRS $(\Sigma_{\mathbb{U}^\lambda} = \Sigma_\mathbb{S} \uplus \Sigma_\mathbb{T}, E')$ is terminating (SN), then

$$\text{congruence by } E_\mathbb{S} \cup E_\mathbb{T} \cup E' = \text{congruence by } E_\mathbb{S} \cup E_\mathbb{T} \cup E_\lambda.$$

Lemma

If R is a set rules of the form $s[t_x/x] \to t[s_y/y]$ such that

- $s[t_x/x]$ *has ST-layers* $(1, 1)$
- $t[s_y/y]$ *has TS-layers* $(*, 1)$
- s_y *is linear*[1] *in* $Z = \{t_x \mid t_x \text{ is a variable}\}$,

then R is terminating.

[1] *Linear* in a TRS sense, i.e. variables appearing *at most* once.

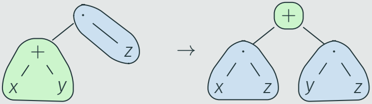
Axiomatisation examples

Example

- Ring from $\lambda : \mathrm{Mon} \cdot \mathrm{AbGrp} \to \mathrm{AbGrp} \cdot \mathrm{Mon}$.

  $(x+y)z = xz + yz$:

  - RHS TS-layers $(1,1)$ ✓
  - linearity ✓



- $\lambda : \mathcal{D}R \to R\mathcal{D}$ (distribution over reader): for each $p \in [0,1]$

$$f(x_1, \ldots, x_n) \oplus_p y = f(x_1 \oplus_p y, \ldots, x_n \oplus_p y).$$

- $\lambda \colon \mathcal{MD} \to \mathcal{DM}$ (multiset over distribution): for each $p \in [0,1]$

$$(x_1 \oplus_p x_2) \cdot y = (x_1 \cdot y) \oplus_p (x_2 \cdot y).$$

- $\lambda \colon \mathrm{Mon}^+\mathrm{Mon}^+ \to \mathrm{Mon}^+\mathrm{Mon}^+$ (non-empty list over itself)

$$a * (b \star c) = a * b$$

$$(a \star b) * c = a * c.$$

Counterexample

Note: $E' \subseteq E_\lambda$ not terminating $\implies$ conclusion not guaranteed.

Example (Famous $ab \to bbaa$ TRS example)

Define two theories and a distributive law:

$$\begin{cases} \Sigma_{\mathbb{S}} := \{a^{(1)}\} \\ E_{\mathbb{S}} := \{aaa = aa\} \end{cases} \qquad \begin{cases} \Sigma_{\mathbb{T}} := \{b^{(1)}\} \\ E_{\mathbb{T}} := \{bbb = bb\} \end{cases}$$

$$\begin{aligned} \lambda\colon ST\mathcal{V} &\to TS\mathcal{V} \\ \overline{a^n\overline{b^m x}^{\mathbb{T}}}^{\mathbb{S}} &\mapsto \overline{b^2\overline{a^2 x}^{\mathbb{S}}}^{\mathbb{T}}, && \text{for } n,m \in \{1,2\} \\ \overline{a^n\overline{x}^{\mathbb{T}}}^{\mathbb{S}} &\mapsto \overline{\overline{a^n x}^{\mathbb{S}}}^{\mathbb{T}}, && \text{for } n \in \{1,2\} \\ \overline{\overline{b^n x}^{\mathbb{T}}}^{\mathbb{S}} &\mapsto \overline{b^n\overline{x}^{\mathbb{S}}}^{\mathbb{T}}, && \text{for } n \in \{1,2\} \\ \overline{\overline{x}^{\mathbb{T}}}^{\mathbb{S}} &\mapsto \overline{\overline{x}^{\mathbb{S}}}^{\mathbb{T}} \end{aligned}$$

However $E' = \{ab = b^2a^2\}$ cannot derive $(aab, bbaa) \in E_\lambda$.

$$a\underline{ab} =_{E'} \underline{ab}baa =_{E'} bba\underline{ab}aa =_{E'} bbabb\underline{aaaa} =_{E_{\mathbb{S}}} bb\underline{ab}baa$$
$$=_{E'} \underline{bbbb}aabaa =_{E_{\mathbb{T}}} bbaabaa =_{E'} \ldots \text{ (loop)}$$

Conclusion

Conclusion

Contribution:

- Proved constructively: Distributive Laws $\iff$ Composite Theories.
  More than the result: it's the proof strategy.
- Gave criteria for minimal axiomatisation $E' \subseteq E_\lambda$.

Future work:

- More TRS criteria for $E' \subseteq E_\lambda$ termination.
- Extend correspondence further:
  - *weak composite theories?*
  - *multi-sorted distributive laws?*