A Categorical Approach to Coalgebraic Fixpoint Logic

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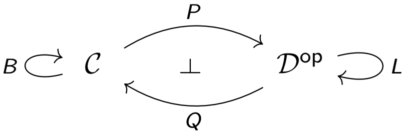
April 6, 2024

Dual Adjunctions



$$\delta : LP \to PB$$

- Generic framework for coalgebraic modal logics
- Can we build on top of it to model coalgebraic fixpoint logics?

Outline

1. ‘Dissect’ a fixpoint logic
2. Formalize what we find

Looking at PDL

$$\alpha, \beta ::= \pi \in \Pi_0 \mid \alpha \cup \beta \mid \alpha ; \beta \mid \mathsf{skip} \mid \mathsf{fail} \mid \alpha^*$$
$$\phi, \psi ::= p \in \mathsf{Prop} \mid \bot \mid \top \mid \phi \wedge \psi \mid \phi \vee \psi \mid \langle \alpha \rangle \phi \mid \neg \phi$$

Unfolding

$$\begin{aligned}
\langle \alpha \cup \beta \rangle \phi &\mapsto \langle \alpha \rangle \phi \vee \langle \beta \rangle \phi \\
\langle \alpha ; \beta \rangle \phi &\mapsto \langle \alpha \rangle \langle \beta \rangle \phi \\
\langle \mathsf{skip} \rangle \phi &\mapsto \phi \\
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\langle \alpha^* \rangle \phi &\mapsto \phi \vee \langle \alpha \rangle \langle \alpha^* \rangle \phi
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Positive logical connections

$$\mathcal{C} \underset{Q}{\overset{P}{\rightleftarrows}} \mathcal{D}^{\mathsf{op}} \quad (\bot)$$

- Enrich $\mathcal{D}$ in **Posets**
- $\mathbf{Hom}_{\mathcal{D}}(\mathbb{A}, PX)$ is a complete lattice

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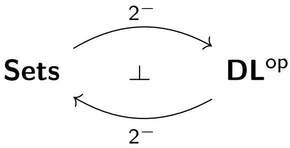

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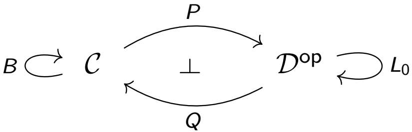

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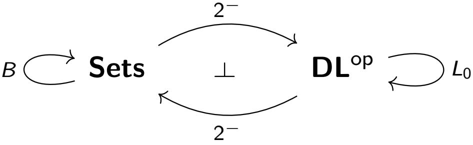

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PDL

$$BX := \mathcal{P}(\mathrm{Prop}) \times \mathcal{P}(X)^{\Pi_0}$$

$$L_0\mathbb{A} := \mathrm{Free}(\mathrm{Prop} \cup \{\langle \pi \rangle a \mid \pi \in \Pi_0, a \in \mathbb{A}\})$$

$$\delta : \begin{cases} p \mapsto \{(m, u) \mid p \in m\} \\ \langle \pi \rangle U \mapsto \{(m, u) \mid u(p) \cap U \neq \varnothing\} \end{cases}$$

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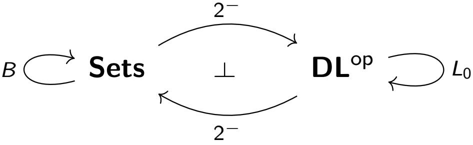

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Staying with PDL: why not take $L : \mathcal{D} \to \mathcal{D}$ as

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This defines a natural transformation

$$L \to \mathsf{Term}(L_0 + L)$$

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We write $\alpha : L\Phi \to \Phi$ for the initial L-algebra (the algebra of formulas)

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Rather than being a unique algebra map, $[\![-]\!]$ is a least solution satisfying

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$$\begin{array}{ccc} \Phi & \xrightarrow{\;[\![-]\!]\;} & PX \\ \Big\downarrow{\scriptstyle \alpha^{-1}} & & \Big\uparrow{\scriptstyle [PX, P\gamma]} \\ L\Phi & & PX + PBX \\ \Big\downarrow{\scriptstyle \mathsf{u}} & & \Big\uparrow{\scriptstyle PX+\delta} \\ \Phi + L_0 L\Phi & & \\ \Big\downarrow{\scriptstyle \mathrm{id}+L_0\alpha} & & \\ \Phi + L_0\Phi & \xrightarrow{\;[\![-]\!]+L_0[\![-]\!]\;} & PX + L_0 PX \end{array}$$

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Thank you!