Coalgebra Dreams

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  Where can this take us?

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I have tried to encourage novel areas of application of coalgebra, e.g.

- SA, Coalgebras, Chu Spaces and Representations of Physical Systems, LiCS 2010.
- SA and J. Zvesper, From Lawvere to Brandenburger-Keisler: interactive forms of diagonalization and self-reference, CMCS 2012 and JCSS 2015.
- SA and V. Winschel, Coalgebraic analysis of subgame-perfect equilibria in infinite games without discounting, MSCS 2017.

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Let's not forget to dream!

Structure *vs* Power: The Great Divide

Structure:

- compositionality, semantics
- How we can master the complexity of computer systems and software?

Power:

- expressiveness, complexity
- How we can harness the power of computation and recognize its limits?

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Mazur quoting Lenstra:

> *twenty years ago he was firm in his conviction that he DID want to solve Diophantine equations, and that he DID NOT wish to represent functors – and now he is amused to discover himself representing functors in order to solve Diophantine equations!*

The topic for this talk

We shall discuss a novel approach to relating categorical semantics, which exemplifies "Structure", to finite model theory, which exemplifies "Power".

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Based on:

- "The Pebbling Comonad in Finite Model Theory", SA, Anuj Dawar and Pengming Wang, LiCS 2017.
- "Relating Structure to Power: comonadic semantics for computational resources", SA and Nihil Shah, extended abstract in CMCS proceedings, to appear in CSL 2018.

and ongoing work with Nihil Shah and Tom Paine.

Model theory and deception

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$$\mathcal{A} \equiv^{\mathcal{L}} \mathcal{B} \quad \overset{\Delta}{\Longleftrightarrow} \quad \forall \varphi \in \mathcal{L}.\, \mathcal{A} \models \varphi \iff \mathcal{B} \models \varphi.$$

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- In descriptive complexity, we seek to characterize a complexity class **C** (for decision problems) as those classes of structures $\mathcal{K}$ (e.g. graphs) definable in $\mathcal{L}$.

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The EF-game between $\mathcal{A}$ and $\mathcal{B}$. In the i'th round, Spoiler moves by choosing an element in A or B; Duplicator responds by choosing an element in the other structure. Duplicator wins after k rounds if the relation $\{(a_i, b_i) \mid 1 \leq i \leq k\}$ is a partial isomorphism.

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Similarly, there are k-pebble games, and bismulation games played to depth k.

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Just as for EF-games, there is an existential-positive version, in which Spoiler only plays in $\mathcal{A}$, and Duplicator responds in $\mathcal{B}$.

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- Thus the notion of local approximation built into the game is internalised into the category of σ-structures and homomorphisms.
- This leads to comonadic and coalgebraic characterisations of a number of central concepts in Finite Model Theory and combinatorics.

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Our setting will be $\mathcal{R}(\sigma)$, the category of relational structures and homomorphisms.

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Given a homomorphism $f : \mathbb{E}_k\mathcal{A} \to \mathcal{B}$, we define the coextension $f^* : A^{\leq k} \to B^{\leq k}$ by

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This is easily verified to yield a comonad on $\mathcal{R}(\sigma)$.

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The winning condition for Duplicator in this game is that, after k rounds have been played, the induced relation $\{(a_i, b_i) \mid 1 \leq i \leq k\}$ is a partial homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

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Spoiler plays only in $\mathcal{A}$, and $b_i = f[a_1, \ldots, a_i]$ represents Duplicator's response in $\mathcal{B}$ to the i'th move by Spoiler.

The winning condition for Duplicator in this game is that, after k rounds have been played, the induced relation $\{(a_i, b_i) \mid 1 \leq i \leq k\}$ is a partial homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

Theorem

The following are equivalent:

1. *There is a homomorphism $\mathbb{E}_k\mathcal{A} \to \mathcal{B}$.*
2. *Duplicator has a winning strategy for the existential Ehrenfeucht-Fraïssé game with k rounds, played from $\mathcal{A}$ to $\mathcal{B}$.*
3. *For every existential positive sentence φ with quantifier rank $\leq k$, $\mathcal{A} \models \varphi \;\Rightarrow\; \mathcal{B} \models \varphi$.*

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Given a structure $\mathcal{A}$, the universe of $\mathbb{P}_k\mathcal{A}$ is $(\mathbf{k} \times A)^+$, the set of finite non-empty sequences of moves (p, a). Note this will be infinite even if $\mathcal{A}$ is finite.
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Given a homomorphism $f : \mathbb{P}_k\mathcal{A} \to \mathcal{B}$, we define the coextension $f^* : \mathbb{P}_k\mathcal{A} \to \mathbb{P}_k\mathcal{B}$ by

$$f^*[(p_1, a_1), \ldots, (p_j, a_j)] = [(p_1, b_1), \ldots, (p_j, b_j)],$$

where $b_i = f[(p_1, a_1), \ldots, (p_i, a_i)]$, $1 \leq i \leq j$.

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The universe of $\mathbb{M}_k(\mathcal{A}, a)$ comprises $[a]$, which is the distinguished element, together with all sequences of the form $[a_0, \alpha_1, a_1, \ldots, \alpha_j, a_j]$, where $a = a_0$, $1 \leq j \leq k$, and $R^{\mathcal{A}}_{\alpha_i}(a_i, a_{i+1})$, $0 \leq i < j$.

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In each case, we write

- $\exists\mathcal{L}_k$ for the existential positive fragment of $\mathcal{L}_k$
- $\mathcal{L}_k(\#)$ for the extension of $\mathcal{L}_k$ with counting quantifiers $\exists_{\leq n}$, $\exists_{\geq n}$

Characterization

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We can generically define two equivalences based on our indexed comonads $\mathbb{E}_k$:

- $\mathcal{A} \rightleftarrows_k^{\mathbb{E}} \mathcal{B}$ iff there are coKleisli morphisms $\mathbb{E}_k\mathcal{A} \to \mathcal{B}$ and $\mathbb{E}_k\mathcal{B} \to \mathcal{A}$. Note that there need be no relationship between these morphisms.
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For details, see the CSL paper.

Characterizing logical equivalences

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Theorem

For structures $\mathcal{A}$ and $\mathcal{B}$:

(1) $\mathcal{A} \equiv^{\exists \mathcal{L}_k} \mathcal{B} \iff \mathcal{A} \rightleftarrows_k \mathcal{B}.$

(2) $\mathcal{A} \equiv^{\mathcal{L}_k} \mathcal{B} \iff \mathcal{A} \leftrightarrow_k \mathcal{B}.$

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Thus in each case, we capture the salient logical equivalences in syntax-free, categorical form.

Coalgebras and combinatorial parameters

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Conceptually, we can think of the morphisms $f : \mathbb{C}_k\mathcal{A} \to \mathcal{B}$ in the co-Kleisli category for $\mathbb{C}_k$ as those which only have to respect the k-local structure of $\mathcal{A}$.

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The lower the value of k, the less information available to Spoiler, and the easier it is for Duplicator to have a winning strategy. Equivalently, the easier it is to have a morphism from $\mathcal{A}$ to $\mathcal{B}$ in the co-Kleisli category.

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Restricting the access to $\mathcal{B}$ makes it *harder* for Duplicator to win the homomorphism game.

Coalgebras: a novel perspective

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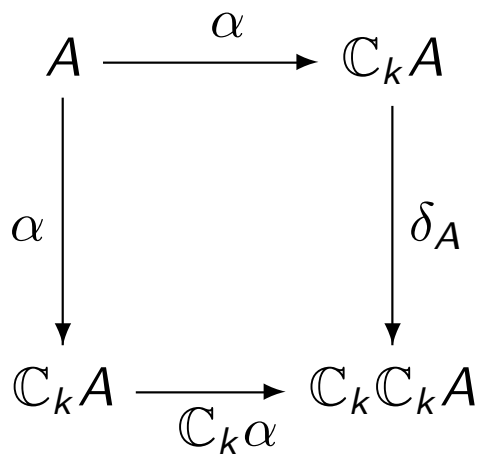
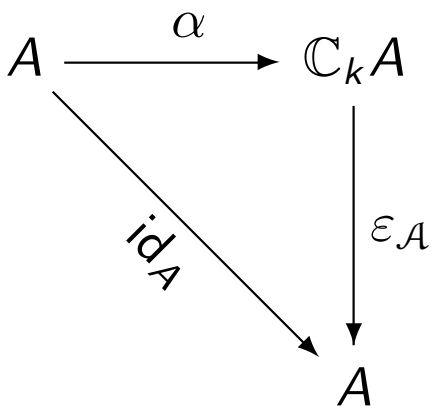
A coalgebra for a comonad (G, ε, δ) is a morphism $\alpha : A \to GA$ such that the following diagrams commute:





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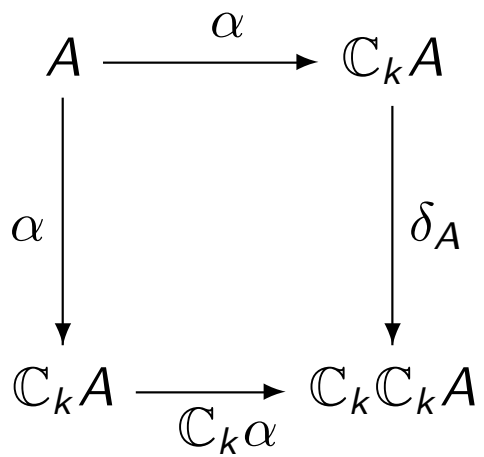

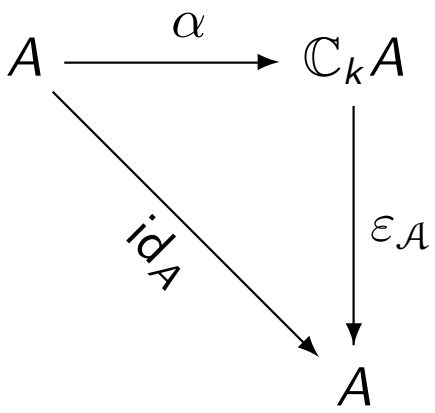

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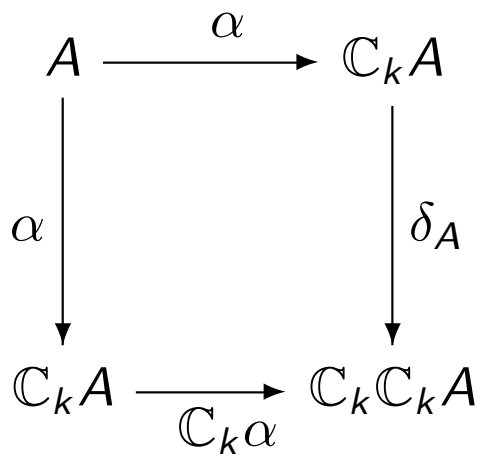

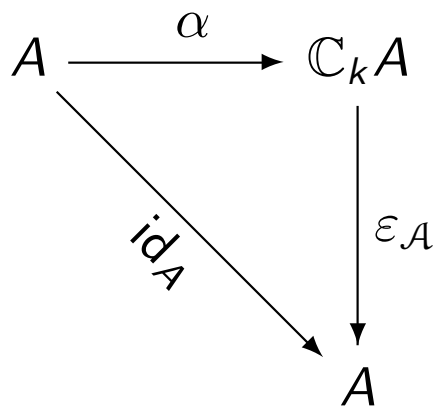

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Our use of indexed comonads $\mathbb{C}_k$ opens up a new kind of question for coalgebras. Given a structure $\mathcal{A}$, we can ask: what is the least value of k such that a $\mathbb{C}_k$-coalgebra exists on $\mathcal{A}$? We call this the *coalgebra number* of $\mathcal{A}$.

Coalgebra numbers

Theorem

- *For the pebbling comonad, the coalgebra number of $\mathcal{A}$ corresponds precisely to the tree-width of $\mathcal{A}$.*
- *For the Ehrenfeucht-Fraïssé comonad, the coalgebra number of $\mathcal{A}$ corresponds precisely to the tree-depth of $\mathcal{A}$.*
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We thus obtain categorical characterizations of these key combinatorial parameters.

Tree depth and the Ehrenfeucht-Fraïssé comonad

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The tree-depth $\mathrm{td}(G)$ is defined to be $\min_F \mathrm{ht}(F)$, where F ranges over forest covers of G.

Tree depth and the Ehrenfeucht-Fraïssé comonad

A *forest* is a poset $(F, \leq)$ such that, for all $x \in F$, the set of predecessors is a finite chain.

A forest cover for G is a forest $(F, \leq)$ such that $V \subseteq F$, and if $v \frown v'$, then $v \uparrow v'$.

The tree-depth $\mathrm{td}(G)$ is defined to be $\min_F \mathrm{ht}(F)$, where F ranges over forest covers of G.

Given a σ-structure $\mathcal{A}$, the Gaifman graph $\mathcal{G}(\mathcal{A})$ is $(A, \frown)$, where $a \frown a'$ iff for some relation $R \in \sigma$, for some $(a_1, \ldots, a_n) \in R^{\mathcal{A}}$, $a = a_i$, $a' = a_j$, $i \neq j$. The tree-depth of $\mathcal{A}$ is $\mathrm{td}(\mathcal{G}(\mathcal{A}))$.

Theorem

Let $\mathcal{A}$ be a finite σ-structure, and $k > 0$. There is a bijective correspondence between

1. *$\mathbb{E}_k$-coalgebras $\alpha : \mathcal{A} \to \mathbb{E}_k\mathcal{A}$.*
2. *Forest covers of $\mathcal{G}(\mathcal{A})$ of height $< k$.*

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Thus $(A, \leq)$ is a forest cover of $\mathcal{A}$, of height $\leq k$.

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A tree-decomposition of a graph $G = (V, \frown)$ is a tree $(T, \leq)$ together with a labelling function $\lambda : T \to \mathcal{P}(V)$ satisfying the following conditions:

- (TD1) for all $v \in V$, for some $x \in T$, $v \in \lambda(x)$;
- (TD2) if $v \frown v'$, then for some $x \in T$, $\{v, v'\} \subseteq \lambda(x)$;
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This parameter plays a fundamental role in combinatorics, algorithms and parameterized complexity.

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A k-pebble forest cover for a graph $G = (V, \frown)$ is a forest cover $(V, \leq)$ together with a pebbling function $p : V \to \mathbf{k}$ such that, if $v \frown v'$ with $v \leq v'$, then for all $w \in (v, v']$, $p(v) \neq p(w)$.

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Theorem

Let G be a finite graph. The following are equivalent:

1. *G has a tree decomposition of width $< k$.*
2. *G has a k-pebble forest cover.*

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Theorem

For all finite structures $\mathcal{A}$: $\mathsf{tw}(\mathcal{A}) = \kappa^{\mathbb{P}}(\mathcal{A}) - 1$.

Further Horizons I

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- It uses colimits to glue existential positive witnesses, in order to reduce elementary equivalence (at given quantifier rank) to existential positive equivalence. (A sort of quantifier elimination).
- Should be generalizable, e.g. to finite variable/pebbling case.

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Major algorithmic metatheorems:

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The wider issue: can we get Structure and Power to work with each other to address genuinely deep questions?

Envoi

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Let's not forget to dream!