The Method of Coalgebra: exercises in coinduction

Jan Rutten

CWI Amsterdam & Radboud University Nijmegen

Summary

- Shameless publicity for a new book
- Sketching its motivation and contents
- As of today, a draft is on-line at:

  https://homepages.cwi.nl/∼janr/papers
- Comments are welcome
- 1 January 2019: final version on-line, freely available at:

  https://homepages.cwi.nl/∼janr/papers

Acknowledgement

I am most grateful to my former PhD students for our joint study of coalgebra, on which much of the present book is based:

1996 Daniele Turi

2004 Falk Bartels

2006 Clemens Kupke

2009 Helle Hansen

2010 David Costa

2010 Alexandra Silva

2013 Georgiana Caltais

2014 Joost Winter

2016 Jurriaan Rot

2018 Henning Basold

2018 Julian Salamanca

Overview of this talk

1. A few words on history
2. The role of categories
3. The notion of homomorphism
4. A table of contents: emphasis on coinduction (up-to)
5. What we do not do
6. Some larger examples of coinduction
7. Where coalgebra is used

1. A few words on history

- Arbib and Manes
- Park and Milner
- Scott, Smyth and Plotkin
- De Bakker and Zucker, America
- Aczel and Mendler
- Abramsky
- Many of you
- 20 years ago: first CMCS in Lisbon, 1998
  Jacobs, Moss, Reichel, Rutten

2. The role of category theory

- No excuses, beating around the bush, apologies, reassurances
- First chapter: *Categories – where coalgebra comes from*
- (But then again: *it may be skipped on first reading.*)

Why categories?

From Samson Abramsky's tutorial:

Categories, why and how?

(Dagstuhl, January 2015)

Why categories?

For **logicians**: gives a syntax-independent view of the fundamental structures of logic, opens up new kinds of models and interpretations.

For **philosophers**: a fresh approach to structuralist foundations of mathematics and science; an alternative to the traditional focus on set theory.

For **computer scientists**: gives a precise handle on abstraction, representation-independence, genericity and more. Gives the fundamental mathematical structures underpinning programming concepts.

Why categories?

For **mathematicians**: organizes your previous mathematical experience in a new and powerful way, reveals new connections and structure, allows you to “think bigger thoughts”.

For **physicists**: new ways of formulating physical theories in a structural form. Recent applications to Quantum Information and Computation.

For **economists and game theorists**: new tools, bringing complex phenomena into the scope of formalisation.

Category Theory in Slogans

- Always ask: what are the **types**?
- Think in terms of **arrows** rather than **elements.**
- Ask what mathematical structures **do**, not what they **are.**
- **Functoriality!**
- **Universality!**
- **Duality!**

  + several others.

All of the above are very relevant for coalgebra.

Always ask: what are the types?

$$A \xrightarrow{f} B \xrightarrow{g} C$$

For instance, for sets and functions:

Not:
let f be a function defined for any x by $f(x) = \ldots$

Rather:
let $f : X \to Y$ be a function defined for any $x \in X$ by $f(x) = \ldots$

Think in terms of arrows rather than elements

An example:

A function $f : X \to Y$ (between sets) is:

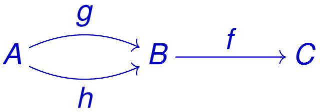
- **injective**: $\forall x, y \in X : \ f(x) = f(y) \Rightarrow x = y$

- **surjective**: $\forall y \in Y \ \exists x \in X : \ f(x) = y$

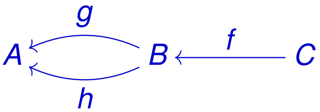
Think in terms of arrows rather than elements

A function $f : X \to Y$ (between sets) is:

- **monic**: $\forall g, h : \quad f \circ g = f \circ h \quad \Rightarrow \quad g = h$



- **epic**: $\forall g, h : \quad g \circ f = h \circ f \quad \Rightarrow \quad g = h$



Think in terms of arrows rather than elements

Proposition

- m is injective iff m is monic.
- e is surjective iff e is epic.

Ask what mathematical structures **do**, not what they **are**

Defining the (Cartesian) **product** of objects A and B ...

- **with elements:**

$$A \times B = \{ \langle a, b \rangle \mid a \in A,\ b \in B \}$$

where

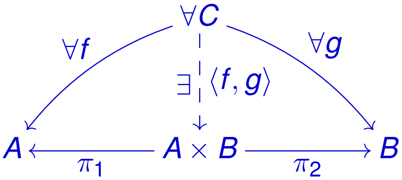
$$\langle a, b \rangle = \{\{a, b\},\ b\}$$

This definition is by no means *canonical*,
does not seem to express any of its *intrinsic properties*,
and feels like *coding*.

Ask what mathematical structures **do**, not what they **are**

Defining the (categorical) **product** of objects A and B . . .

- **with arrows** (expressing a **universal** property):



This defines the **behaviour** of the product
by specifying its **interactions** with other objects.

Functoriality!

A *functor* $F : \mathcal{C} \to \mathcal{D}$ maps:

(i) each **object** A in $\mathcal{C}$ to an **object** $F(A)$ in $\mathcal{D}$

(ii) each **arrow** $f : A \to B$ in $\mathcal{C}$ to
an **arrow** $F(f) : F(A) \to F(B)$ in $\mathcal{D}$

such that $F(g \circ f) = F(g) \circ F(f)$ and $F(id_A) = id_{F(A)}$

E.g., the *powerset functor* $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ maps sets X to

$$\mathcal{P}(X) = \{V \mid V \subseteq X\}$$

and functions $f : X \to Y$ to

$$\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y) \qquad V \mapsto \{f(v) \mid v \in V\}$$

Functoriality!

Is just natural since all we have are objects and arrows.

Will be crucial for the definition of homomorphism between algebras and coalgebras.

Universality!

Ideally, definitions are phrased in terms of *universal properties*, which are typically formulated as:

$$\forall \;\ldots\; \exists! \;\ldots$$

E.g., an object A in a category $\mathcal{C}$ is **initial** if:
for any object B in $\mathcal{C}$ there exists a unique arrow from A to B:

$$\forall\, B \xleftarrow{\;\exists!\;} A$$

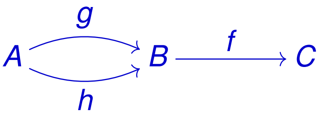
Similarly, an object A is **final** if:
for any object B in $\mathcal{C}$ there exists a unique arrow from B to A:

$$\forall\, B \xrightarrow{\;\exists!\;} A$$

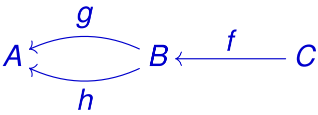
Duality!

Informally, duality refers to the elementary process of "reversing the arrows" in a diagram.

E.g., f is **monic**: $\forall g, h: \quad f \circ g = f \circ h \Rightarrow g = h$



Reversing the arrows: $\forall g, h: \quad g \circ f = h \circ f \Rightarrow g = h$



That is, f is **epic.**

Duality, formally

The *opposite* category $\mathcal{C}^{op}$ of a category $\mathcal{C}$ has:

- the same objects as $\mathcal{C}$
- precisely one arrow $f : B \to A$ for every arrow $f : A \to B$ in $\mathcal{C}$.

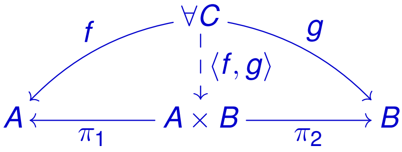
The principle of duality now says that we can dualize any statement about a category $\mathcal{C}$ by making the same statement about $\mathcal{C}^{op}$.

For instance, the notions of monic and epic are dual, since:

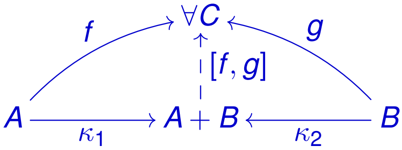
Proposition: f is monic in $\mathcal{C}$ iff f is epic in $\mathcal{C}^{op}$.

Duality: products and coproducts

The **product** of A and B:



The **coproduct** of A and B:



Duality: initial and final objects

An object A in a category $\mathcal{C}$ is $\ldots$

- **initial** if for any object B there exists a unique arrow

$$A \overset{!}{\dashrightarrow} B$$

- **final** if for any object B there exists a unique arrow

$$B \overset{!}{\dashrightarrow} A$$

Algebra, categorically

For a functor $F : \mathcal{C} \to \mathcal{C}$, an F**-algebra** is a pair (A, α) with

$$\begin{array}{c} F(X) \\ \alpha \big\downarrow \\ X \end{array}$$

We call F the **type** and α the **structure map** of (A, α).

Coalgebra, dually

For a functor $F : \mathcal{C} \to \mathcal{C}$, an F-**coalgebra** is a pair (A, α) with

$$X \xrightarrow{\alpha} F(X)$$

We call F the **type** and α the **structure map** of (A, α).

3. The notion of homomorphism

- Cf. *structure preserving maps* between dynamical systems

  F.W. Lawvere and S.H. Schanuel.
  Conceptual mathematics.
  Cambridge University Press, 1997.

Homomorphisms of dynamical systems

$$
\begin{array}{ccc}
S & \xrightarrow{f} & T \\
\alpha \big\downarrow & & \big\downarrow \beta \\
S & \xrightarrow[f]{} & T
\end{array}
$$

One way of seeing that the notion of homomorphism of dynamical systems makes sense, is to note that isomorphic dynamical systems

$$(S, \alpha) \;\cong\; (T, \beta)$$

have essentially the same dynamics*: they have not only the same number of states but also equal numbers of fixed points, the same number of cycles of length seven, equal numbers of states that move nine steps before entering a cycle of length four, etc. etc.*

The behaviour of dynamical systems

$$\begin{array}{ccc} S & \xrightarrow{\exists!} & \mathbf{1} \\ \alpha\downarrow & & \downarrow = \\ S & \longrightarrow & \mathbf{1} \end{array}$$

All states are bisimilar: too early to discuss bisimulations.
They will be introduced with stream systems.

However, homomorphisms of dynamical systems are definitively interesting.

Homomorphisms of dynamical systems

In the remainder of this chapter, we will illustrate how one can learn more about (the internal structure of) a dynamical system (S, α) *by* **probing it *from the outside* with homomorphisms**.

Exercise: Let $n \geqslant 1$ and consider the following dynamical system:

$$(C_n, \gamma_n) \quad = \quad 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \quad (n-1 \to 0)$$

Let (S, α) be an arbitrary dynamical system. Let H be the following set of homomorphisms:

$$H \;=\; \{ f \mid f\colon (C_n, \gamma_n) \to (S, \alpha) \}$$

Show that H is isomorphic to the set of all points of (S, α) with period n.

4. A table of contents

Table of contents

1. Introduction
2. Categories – where coalgebra comes from
3. Algebras and coalgebras
4. Induction and coinduction
5. The method of coalgebra
6. Dynamical systems
7. Stream systems
8. Deterministic automata
9. Partial automata
10. Non-deterministic automata
11. Stream differential equations
12. A calculus of streams
13. Mealy automata
14. Weighted stream automata
15. Universal coalgebra
16. Notation and preliminaries

5. What we do not do

- modal and coalgebraic logic
- Haskell or other programming languages
- universal coalgebra (or: hardly)
- distributive laws
- general account of coinduction up-to

There is room for more (and better) books.

6. Some larger examples of coinduction

(i) Moessner's theorem
(ii) languages and automata
(iii) synthesis of Mealy machines for bitstream functions

Moessner's Theorem (k = 2)

nat	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$Drop_2$	1		3		5		7		9		11	$\cdots$	
Σ	1	4	9	16	25	36	$\cdots$						
	$=$												
nat^2	1^2	2^2	3^2	4^2	5^2	6^2	$\cdots$						

Moessner's Theorem (k = 3)

nat	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$Drop_3$	1	2		4	5		7	8		10	11	$\cdots$	
Σ	1	3	7	12	19	27	37	48	$\cdots$				
$Drop_2$	1		7		19		37		$\cdots$				
Σ	1	8	27	64	$\cdots$								
	$=$												
nat^3	1^3	2^3	3^3	4^3	$\cdots$								

Moessner's Theorem (k = 4)

nat	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$Drop_4$	1	2	3		5	6	7		9	10	11	$\cdots$
Σ	1	3	6	11	17	24	33	43	54	$\cdots$		
$Drop_3$	1	3		11	17		33	43		67	81	$\cdots$
Σ	1	4	15	32	65	108	175	$\cdots$				
$Drop_2$	1		15		65		175	$\cdots$				
Σ	1	16	81	256	$\cdots$							
	$=$	1^4	2^4	3^4	4^4	$\cdots$						

Formalising Moessner's theorem (k = 3)

$$
\begin{aligned}
\mathrm{nat}^3 &= \Sigma \circ D_2 \circ \Sigma \circ D_3\,(\mathrm{nat}) \\
&= \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1})
\end{aligned}
$$

A proof by coinduction

$$\mathrm{nat}^3 = \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\bar{1})$$

The aim is to construct a **bisimulation** relation containing the pair

$$\langle\ \mathrm{nat}^3\,,\ \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\bar{1})\ \rangle$$

Towards that end, let us investigate
the **derivatives** of the streams and operators above.

(**Initial values** will all be straightforward.)

Inspecting derivatives

For the stream $\mathsf{nat} = (1, 2, 3, \ldots)$, we have

$$\begin{aligned} \mathsf{nat}' &= (2, 3, 4, \ldots) \\ &= (1+1,\ 1+2,\ 1+3,\ \ldots) \\ &= (1, 1, 1, \ldots) \oplus (1, 2, 3, \ldots) \\ &= \overline{1} \oplus \mathsf{nat} \end{aligned}$$

where $\oplus$ denotes the elementwise sum of streams.

Inspecting derivatives

For the product $\sigma \odot \tau$, we have

$$(\sigma \odot \tau)' = \sigma' \odot \tau'$$

Inspecting derivatives

These properties of nat' and $(\sigma \odot \tau)'$ imply:

$$
\begin{aligned}
(\mathsf{nat}^3)' &= (\mathsf{nat}')^3 \\
&= (\overline{1} \oplus \mathsf{nat})^3 \\
&= \binom{3}{0} \cdot \overline{1} \;\oplus\; \binom{3}{1} \cdot \mathsf{nat} \;\oplus\; \binom{3}{2} \cdot \mathsf{nat}^2 \;\oplus\; \binom{3}{3} \cdot \mathsf{nat}^3
\end{aligned}
$$

Inspecting derivatives

$$\mathrm{nat}^3 = \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1})$$

So for the stream on the left, we have:

$$(\mathrm{nat}^3)' = \binom{3}{0} \cdot \overline{1} \;\oplus\; \binom{3}{1} \cdot \mathrm{nat} \;\oplus\; \binom{3}{2} \cdot \mathrm{nat}^2 \;\oplus\; \binom{3}{3} \cdot \mathrm{nat}^3$$

Inspecting derivatives

$$\mathrm{nat}^3 = \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1})$$

Turning to the right hand side, we observe:

$$\overline{1}' = \overline{1}$$

Inspecting derivatives

For the drop operators, we have

$$(D_2\,\sigma)^{(1)} = D_2\left(\sigma^{(2)}\right)$$
$$(D_3\,\sigma)^{(2)} = D_3\left(\sigma^{(3)}\right)$$
$$(D_4\,\sigma)^{(3)} = D_4\left(\sigma^{(4)}\right)$$

where the repeated derivatives are defined as usual:

$$\sigma^{(0)} = \sigma$$
$$\sigma^{(k+1)} = (\sigma^{(k)})'$$

Inspecting derivatives

$$(\Sigma\,\sigma)' = \overline{\sigma(0)} \;\oplus\; \Sigma\,(\sigma')$$

where

$$\overline{\sigma(0)} = (\sigma(0), \sigma(0), \sigma(0), \ldots)$$

Inspecting derivatives

Together, these properties imply:

$$
\begin{aligned}
&(\ \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1})\ )' \\
&= \binom{3}{0} \cdot\ \overline{1} \\
&\oplus\ \binom{3}{1} \cdot\ \Sigma \circ D_2\,(\overline{1}) \\
&\oplus\ \binom{3}{2} \cdot\ \Sigma \circ D_2 \circ \Sigma \circ D_3\,(\overline{1}) \\
&\oplus\ \binom{3}{3} \cdot\ \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1})
\end{aligned}
$$

(The details would fill 1 or 2 additional slides.)

Proving Moessner's theorem (k = 3)

$$\text{nat}^3 = \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1})$$

All in all, we have found:

$$(\text{nat}^3)' \qquad\qquad (\,\Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1})\,)'$$

$$= \binom{3}{0} \cdot \overline{1} \qquad\qquad = \binom{3}{0} \cdot \overline{1}$$

$$\oplus \binom{3}{1} \cdot \text{nat} \qquad\qquad \oplus \binom{3}{1} \cdot \Sigma \circ D_2\,(\overline{1})$$

$$\oplus \binom{3}{2} \cdot \text{nat}^2 \qquad\qquad \oplus \binom{3}{2} \cdot \Sigma \circ D_2 \circ \Sigma \circ D_3\,(\overline{1})$$

$$\oplus \binom{3}{3} \cdot \text{nat}^3 \qquad\qquad \oplus \binom{3}{3} \cdot \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1})$$

Proving Moessner's theorem (k = 3)

$$\text{nat}^3 = \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1})$$

All in all, we have found:

$(\text{nat}^3)'$	$(\,\Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1})\,)'$	M3′
$= \binom{3}{0} \cdot \overline{1}$	$= \binom{3}{0} \cdot \overline{1}$	M0
$\oplus \binom{3}{1} \cdot \text{nat}$	$\oplus \binom{3}{1} \cdot \Sigma \circ D_2\,(\overline{1})$	M1
$\oplus \binom{3}{2} \cdot \text{nat}^2$	$\oplus \binom{3}{2} \cdot \Sigma \circ D_2 \circ \Sigma \circ D_3\,(\overline{1})$	M2
$\oplus \binom{3}{3} \cdot \text{nat}^3$	$\oplus \binom{3}{3} \cdot \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1})$	M3

Moessner's theorem: the general case

$$\mathrm{nat}^k = \Sigma \circ D_2 \circ \cdots \circ \Sigma \circ D_{k+1}\,(\overline{1})$$

$$\begin{array}{lll}
(\mathrm{nat}^k)' & (\,\Sigma \circ D_2 \circ \cdots \circ \Sigma \circ D_{k+1}\,(\overline{1})\,)' & \mathrm{M}k' \\
= \binom{k}{0} \cdot \overline{1} & = \binom{k}{0} \cdot \overline{1} & \mathrm{M0} \\
\oplus \binom{k}{1} \cdot \mathrm{nat}^1 & \oplus \binom{k}{1} \cdot \Sigma \circ D_2\,(\overline{1}) & \mathrm{M1} \\
\oplus \binom{k}{2} \cdot \mathrm{nat}^2 & \oplus \binom{k}{2} \cdot \Sigma \circ D_2 \circ \Sigma \circ D_3\,(\overline{1}) & \mathrm{M2} \\
\oplus \cdots & \oplus \cdots & \cdots \\
\oplus \binom{k}{k} \cdot \mathrm{nat}^k & \oplus \binom{k}{k} \cdot \Sigma \circ D_2 \circ \cdots \circ \Sigma \circ D_{k+1}\,(\overline{1}) & \mathrm{M}k
\end{array}$$

Moessner's theorem: the general case

And so we define $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ by

$$R = \{ \langle \, nat^k \, , \; \Sigma \circ D_2 \circ \cdots \circ \Sigma \circ D_{k+1} \, (\bar{1}) \, \rangle \mid k \geq 0 \}$$

Is R a **bisimulation relation**?

No, but almost: R is a bisimulation relation **up to sum**!

Bisimulations up to sum

A relation $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ is a bisimulation relation **up to sum** if, for all $(\sigma, \tau) \in R$,

(i) if $(\sigma, \tau) \in R$ then $\sigma(0) = \tau(0)$

(ii) we can write

$$\sigma' = n_1 \cdot \sigma_1 \quad \oplus \quad \cdots \quad \oplus \quad n_r \cdot \sigma_r$$
$$\tau' = n_1 \cdot \tau_1 \quad \oplus \quad \cdots \quad \oplus \quad n_r \cdot \tau_r$$

such that

$$(\sigma_1, \tau_1) \in R \;, \;\; \ldots \;\; , \; (\sigma_r, \tau_r) \in R$$

Coinduction up to sum

Theorem

If $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ is a bisimulation **up to sum**, then

$$\forall \sigma, \tau \in \mathbb{N}^\omega : \quad (\sigma, \tau) \in R \;\; \Rightarrow \quad \sigma = \tau$$

Moessner's theorem: the general case

$$R = \{ \langle \mathrm{nat}^k , \ \Sigma \circ D_2 \circ \cdots \circ \Sigma \circ D_{k+1} (\bar{1}) \rangle \mid k \geq 0 \}$$

is a bisimulation up to sum.

It follows by coinduction up to sum that

$$\mathrm{nat}^k = \Sigma \circ D_2 \circ \cdots \circ \Sigma \circ D_{k+1} (\bar{1})$$

for all $k \geq 0$.

$\square$

The heart of the matter: circularity

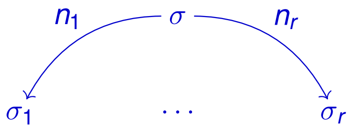
Derivatives in a picture

$$\sigma \longrightarrow \sigma' \longrightarrow \sigma^{(2)} \longrightarrow \sigma^{(3)} \longrightarrow \cdots$$

More generally, if

$$\sigma' = n_1 \cdot \sigma_1 \oplus \ \cdots \ \oplus n_r \cdot \sigma_r$$

then we will write



Circularity

Since

$$\overline{1}' = (1, 1, 1, \ldots)' = \overline{1}$$

we write:

$$\overline{1} \longrightarrow \overline{1} \longrightarrow \overline{1} \longrightarrow \cdots$$

or, equivalently,

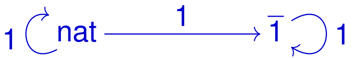
$$\overline{1} \circlearrowleft$$

Circularity

For the stream $\mathsf{nat} = (1, 2, 3, \ldots)$, we have

$$\begin{aligned} \mathsf{nat}' &= (2, 3, 4, \ldots) \\ &= \overline{1} \oplus \mathsf{nat} \qquad\qquad \text{(algebra and coalgebra!)} \end{aligned}$$

$$\mathsf{nat} \longrightarrow \overline{1} \oplus \mathsf{nat} \longrightarrow \overline{1} \oplus \overline{1} \oplus \mathsf{nat} \longrightarrow \cdots$$

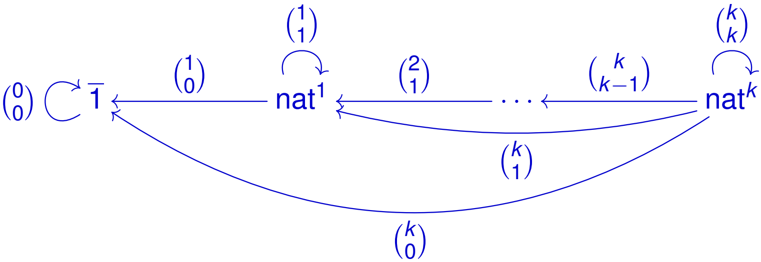


Circularity

Since

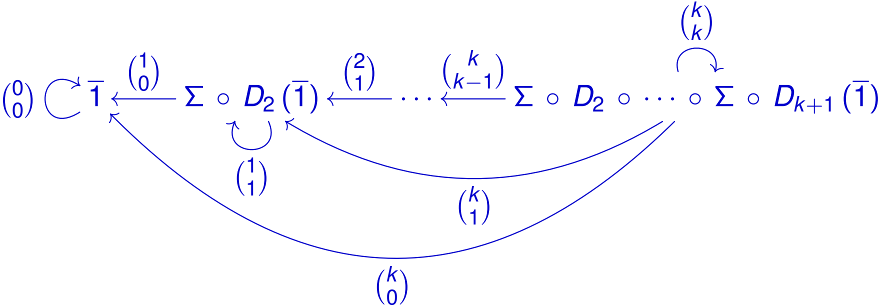
$$(\mathsf{nat}^k)' = \binom{k}{0}\cdot\overline{1} \oplus \binom{k}{1}\cdot\mathsf{nat}^1 \oplus\cdots\oplus \binom{k}{k}\cdot\mathsf{nat}^k$$

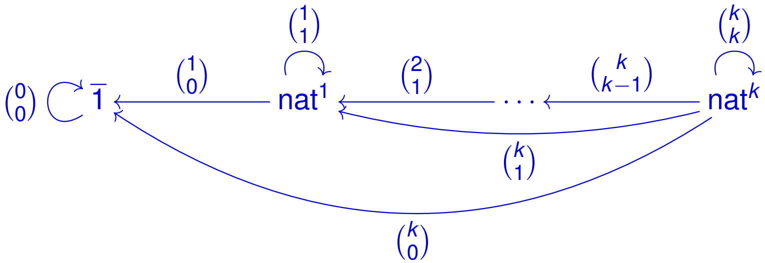
we have

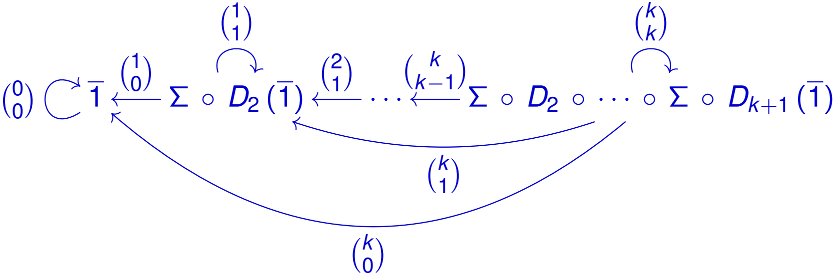


Circularity

Similarly, we have found




$\binom{0}{0}$
$\overline{1}$
$\binom{1}{0}$
$\binom{1}{1}$
nat^1
$\binom{2}{1}$
$\cdots$
$\binom{k}{k-1}$
$\binom{k}{k}$
nat^k
$\binom{k}{1}$
$\binom{k}{0}$

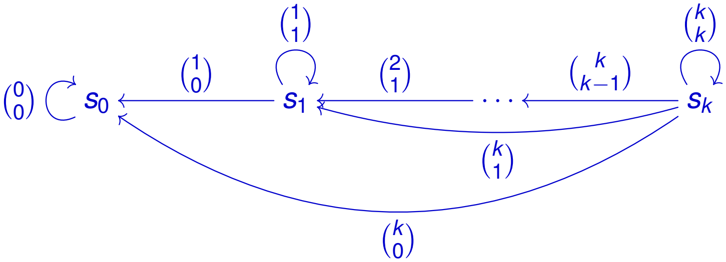

$\binom{0}{0}$
$\overline{1}$
$\binom{1}{0}$
$\binom{1}{1}$
$\Sigma \circ D_2(\overline{1})$
$\binom{2}{1}$
$\cdots$
$\binom{k}{k-1}$
$\binom{k}{k}$
$\Sigma \circ D_2 \circ \cdots \circ \Sigma \circ D_{k+1}(\overline{1})$
$\binom{k}{1}$
$\binom{k}{0}$

The proof of Moessner, in other words

$$\mathsf{nat}^k \;=\; \Sigma \circ D_2 \circ \cdots \circ \Sigma \circ D_{k+1}\,(\overline{1})$$

Both streams **are** the same ...

because they **behave** the same (as the state s_k below):

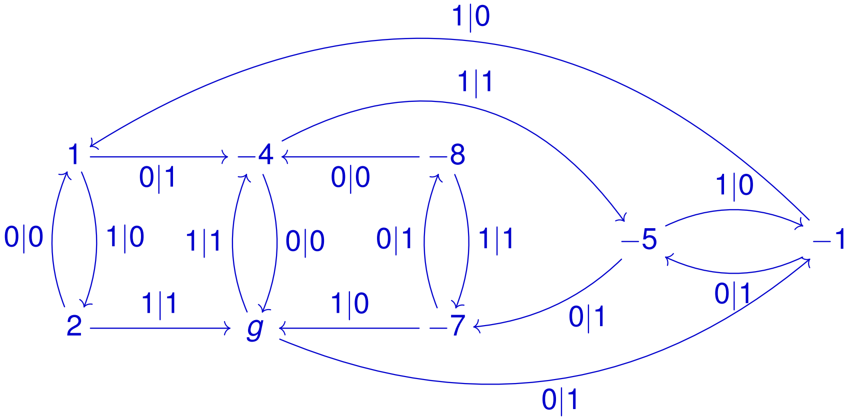


6. Some larger examples of coinduction

(i) Moessner's theorem
(ii) languages and automata
(iii) synthesis of Mealy machines for bitstream functions

Mealy machines for functions on bitstreams

$$g\colon 2^\omega \to 2^\omega \qquad\qquad g(\alpha) = \frac{-2 + (3 \times \alpha)}{9}$$



7. Where coalgebra is used

- logic, set theory
- automata
- control theory
- combinatorics
- data types
- dynamical systems
- games
- economy

7. Where coalgebra is used

- Kabbalah
  - Physarum Polycephalum Syllogistic L-Systems and Judaic Roots of Unconventional Computing

    A. Schumann

    Studies in Logic, Grammar and Rhetoric, 2016
  - Abstract:
    We show that in Kabbalah, the esoteric teaching of Judaism, there were developed ideas of unconventional automata in which ...

7. Where coalgebra is used

- robotica
  - On the relationship between bisimulation and combinatorial filter reduction

    H. Rahmani and J. O'Kane

    IEEE International Conference on Robotics and Automation, 2018

- ecology
  - Bridging Disciplinary Gaps in Studies of Human-Environment Relations: A Modelling Framework

    Hauhs, Trancon y Widemann, Klute

    In: Modern Africa: Politics, History and Society, 2018

7. Where coalgebra is used

- Swarm behaviour
  - Context-Based Games of Swarms

    Andrew Schumann

    In: Behaviourism in Studying Swarms: Logical Models of Sensing and Motoring, Springer, 2018.